The *J* Integral

For a crack in an elastic body subject to a load, the elastic energy stored in the body is a function of two independent variables: the displacement of the load, and the area of the crack. The energy release rate is defined by the partial derivative of the elastic energy of the body with respect to the area of the crack.

This definition of the energy release rate assumes that the body is elastic, but invokes no field theory. Indeed, the energy release rate can be determined experimentally by measuring the load-displacement curves of identically loaded bodies with different areas of the cracks. No field need be measured.

Many materials, however, can be modeled with a field theory of elasticity. When a material is modeled by such a field theory, the energy release rate can be represented in terms of field variables by an integral, the *J* integral.

This lecture describes the *J* integral, along with examples of calculation. Uses of the *J* integral are often better appreciated in the context of individual applications, which we will describe in later lectures.

The *J* integral can be developed for both linear and nonlinear elastic theories. The nonlinear elastic theory will be used in class, and the linear elastic theory will be used in a homework problem.

Nonlinear field theory of elasticity. The main ingredients of the nonlinear elastic theory are as follows (http://imechanica.org/node/7794).

- Model a body by a field of material particles.
- Name each material particle by using its coordinate $\mathbf{X}$ when the body is undeformed.
- When the body deforms, the material particle named $\mathbf{X}$ moves to a place of coordinate $\mathbf{x}$.
- Describe the deformation of the entire body by the function $\mathbf{x}(\mathbf{X})$.
- Define the deformation gradient as $F_{iK} = \partial x_i(\mathbf{X}) / \partial X_K$.
- Let $\mathbf{N}$ be the unit vector normal to an element of a surface in the undeformed body.
- Define the nominal traction $\mathbf{T}$ by the force acting on the element of the surface in the deformed body divided by the area of the element in the undeformed body.
- Consider a cubic element in the undeformed body. In the deformed body, the cube changes to some other shape. Define the nominal stress s_{iK} as the force on a face in the deformed body divided by the area of the face in the undeformed body.
- The nominal traction relates to the nominal stress as $T_i = s_{iK} N_K$.
- The field of stress satisfies the condition of equilibrium: $\partial s_{iK}(\mathbf{X}) / \partial X_K = 0$.
- The energy of the body is $U = \int W dV$. The energy of the body is the sum of energy of all material particles. The energy density is of the body in the undeformed body, but the integration is carried over the volume of the undeformed body.
- A material model is specified by an energy-density function $W = W(\mathbf{F})$.
- The nominal stress relates to the deformation gradient as $s_{iK} = \partial W(\mathbf{F}) / \partial F_{iK}$.

Energy release rate. Here we summarize essential ideas developed in a previous lecture (http://imechanica.org/node/7773). We assume that the body deforms under the plane-strain conditions. For a crack in an elastic body subject to a load, we regard the elastic energy of the body, *U*, as a function of two independent variables:

$$U = U(\Delta, a),$$

where Δ is the displacement of the load, and *a* the length of the crack. The area of the crack is measured in the undeformed body. The function $U(\Delta, a)$ can be determined by two alternative methods:

1. *Experimental method*. Measure the load-displacement curve for a body containing a pre-cut crack. During the measurement, the crack does not extend. Integrate the load-displacement curve to obtain the elastic energy for the body with the specific length of crack. Repeat the procedure for the body with a pre-cut crack of a different length. This method invokes no field theory.
2. *Computational method*. Solve the boundary-value problem for a body containing a pre-cut crack of a given length. Calculate the energy density at every material particle in the body. Integrate the energy density overall all material particles to obtain the elastic energy for the body with the specific area of crack. Repeat the procedure for the body with a pre-cut crack of a different lengh. This method requires that the body be modeled by a field theory of elasticity.

The energy release rate is defined as

$$G = -\frac{\partial U(\Delta, a)}{\partial a}.$$

The energy release rate can also be determined by the two alternative methods.

The potential energy is defined as

$$\Pi = U - P\Delta.$$

The potential energy gives an equivalent definition of the energy release rate:

$$G = -\frac{\partial \Pi(P, a)}{\partial a}.$$

Energy release rate in terms of field variables. Model a body with the nonlinear elastic theory. Under the plane-strain conditions, the body is represented by a region in the plane, and the boundary of the body by a curve in the plane. Each point on the boundary is prescribed by either traction or displacement. The field of deformation in the body is described by the function

$$\mathbf{x} = \mathbf{x}(\mathbf{X}).$$

The potential energy of the body can be calculated by using the field variables:

$$\Pi = \int W(\mathbf{F}) dA - \int T_i(\mathbf{X}) x_i(\mathbf{X}) dL$$

The first integral extends over the body, and the second integral extends over the boundary where traction is prescribed. The potential energy is a functional of the field $\mathbf{x}(\mathbf{X})$.

When the body contains crack, the field of deformation also depends on the length of the crack, namely,

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, a).$$

For two identically loaded bodies with cracks of different lengths, the fields of deformation in the two bodies will be different. Consequently, the potential energy is a function of the length of the crack, $\Pi(a)$, when the load is kept fixed. By definition, the energy release rate is

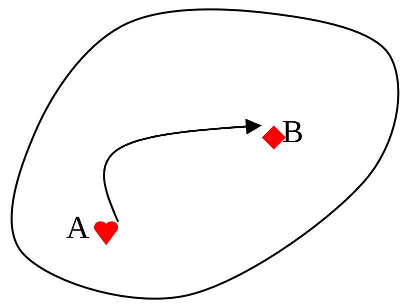
$$G = -\frac{\partial \Pi}{\partial a}.$$

The partial derivative means that we vary the length of the crack, but keep the loading conditions fixed.

The *J* integral. Define a line integral:

$$J = \int \left[W N_1 - N_K \frac{\partial W(\mathbf{F})}{\partial F_{iK}} \frac{\partial x_i(\mathbf{X})}{\partial X_1} \right] dL .$$

The integral is a functional of $\mathbf{x}(\mathbf{X})$, the field that describes the deformation of the body. The dummy variable of integration is $\mathbf{X}$, the coordinates of material particles in the undeformed body. The path of integration is a curve connecting two material particles. The unit vector $\mathbf{N}$ is normal to the line of integration.



The above definition is in terms of the two functions $\mathbf{x}(\mathbf{X})$ and $W(\mathbf{F})$. The integral can also be written in terms of other field variables, e.g.,

$$J = \int \left[W N_1 - T_i F_{i1} \right] dL .$$

In doing so, we use the relations that connect various field variables in the nonlinear elastic theory.

The *J* integral is path-independent. The two material particles A and B can be connected by numerous paths. For any two paths, so long as no singularity exists between them, the *J* integrals along the two paths are identical.

To prove this statement, reverse the direction of one path, so that the two paths form a closed contour. This closed contour is used as a path of integration. We need to prove that the *J* integral vanishes if the closed contour encloses no singularity. The proof invokes the divergence theorem, as follows.

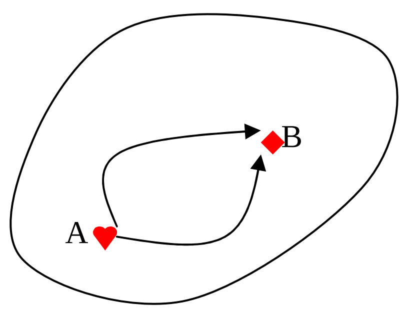
$$\int W N_1 dL = \int \frac{\partial W(\mathbf{F})}{\partial X_1} dA$$

$$= \int \frac{\partial W(\mathbf{F})}{\partial F_{iK}} \frac{\partial F_{iK}(\mathbf{X})}{\partial X_1} dA$$

$$= \int \frac{\partial W(\mathbf{F})}{\partial F_{iK}} \frac{\partial^2 x_i(\mathbf{X})}{\partial X_1 \partial X_K} dA$$

$$= \int \frac{\partial}{\partial X_K} \left[\frac{\partial W(\mathbf{F})}{\partial F_{iK}} \frac{\partial x_i(\mathbf{X})}{\partial X_1} \right] dA$$

$$= \int N_K \frac{\partial W(\mathbf{F})}{\partial F_{iK}} \frac{\partial x_i(\mathbf{X})}{\partial X_1} dL$$



Layered material. In the above, the material is taken to be homogeneous, so that the energy density varies from one material particle to another only through the deformation gradient:

$$W = W(\mathbf{F}).$$

That is, the function does not depend on $\mathbf{X}$. When the two material particles are of the same deformation gradient, they have the same free energy function.

For a layered material, which is homogenous in the X_1-direction, but inhomogeneous in the X_2-direction, the energy density takes the form

$$W = W(\mathbf{F}, X_2).$$

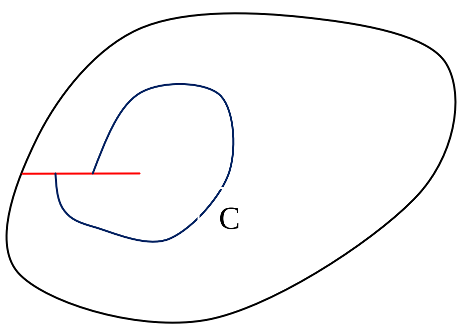
Going through the same steps, you can confirm that the J integral is still path-independent.

Finally consider a generally inhomogeneous material, for which the energy density takes the form

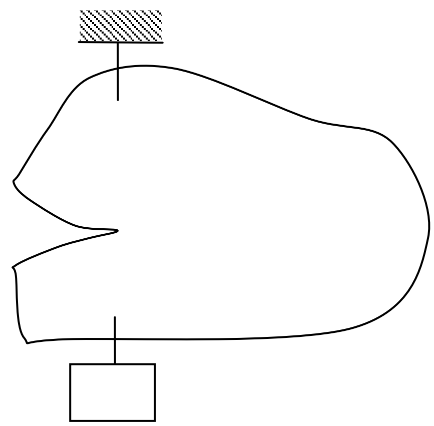
$$W = W(\mathbf{F}, X_1, X_2).$$

Going through the same steps, you will find that the J integral is path-dependent.

The *J* integral equals the energy release rate. Consider a crack in a body subject to a load. The faces of the crack are traction free. In the undeformed body, a path of integration C starts from a material particle on the lower face of the crack, surrounds the tip of the crack, and ends at a material particle on the upper face of the crack. If the path encloses no singularity other than the tip of the crack, the J integral over the path equals the energy release rate.



Undeformed body



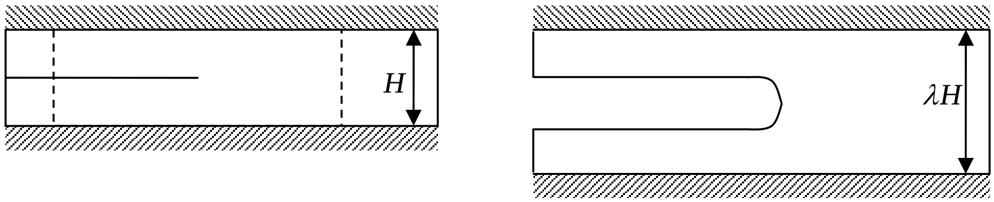
Deformed body

This basic result was described in the original papers on the line integral listed at the end of the notes. You may study these papers for inspiration. I am unaware of any transparent proof of this basic result.

A long crack in a strip of a material pulled by two rigid grips. This test-piece was discussed by Rivlin and Thomas (1953). They obtained the energy release rate directly from its definition:

$$G = HW(\lambda),$$

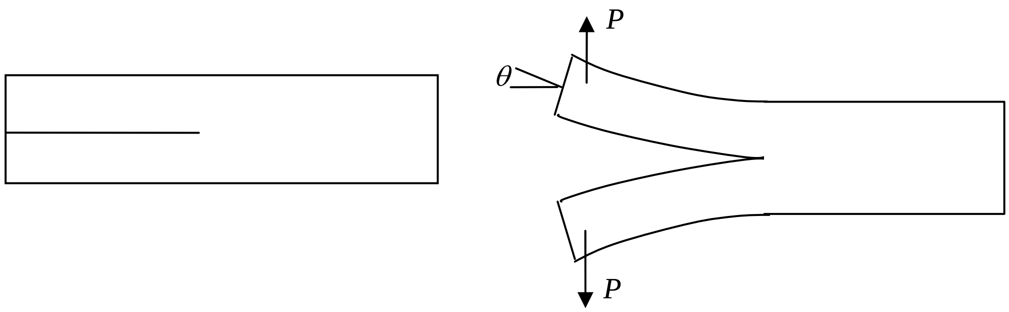
where H is the width of the undeformed strip, and λH is the width of the deformed stripe. The quantity $HW(\lambda)$ can be determined from experimentally measured load-displacement curve of a strip with no crack, pulled by two rigid grips.



The same result can be obtained by using the J integral along the dotted lines indicated in the undeformed strip (Rice, 1968). The only nonzero contribution comes from the vertical line in the strip ahead of the crack.

Double cantilever beams. We have calculated the energy release rate for the double cantilever beam of a linear elastic material. Here is a result for any nonlinear elastic material.

- A.J. Paris and P.C. Paris, Instantaneous evaluation of J and C*. International Journal of Fracture 38, R19-R21 (1988).
- T. Andersson and U. Stigh The stress-elongation relation for an adhesive layer loaded in peel using equilibrium energetic forces. International Journal of Solids and Structures 41, 413-434 (2004).



Take the path to be the external boundary. The only nonzero contribution comes from the vertical boundary far behind the front of crack. The deformation gradient is given by the slope of the beam:

$$\frac{\partial x_2}{\partial X_1} = \theta .$$

The J integral is given by

$$J = 2P\theta .$$

This result is valid even when the separation of the crack is modeled with a nonlinear spring, and is used to experimentally determine the law of separation (Andersson and Stigh, 2004).

Use the *J* integral to calculate energy release rate in the finite-element method. For a crack in an elastic body subject to a load, the energy release rate is defined as the decrease in the elastic energy of the body associated with extension of the crack by unit area, while the load is held fixed. We have used this definition directly to calculate the energy release rate for a few cases. In each case, we somehow obtain the elastic energy in the body as a function of the area of the crack, and then take the partial derivative of this function with respect to the area of the crack.

This procedure, however, is difficult to apply when we solve the boundary-value problem by using the finite-element method. Let us imagine how we might go about it. For a given area of the crack, we use the finite-element method to determine the field in the body, and then integrate to obtain the elastic energy stored in the body. We then change the area of the crack slightly, solve a new boundary-value problem, and then obtain the elastic energy in the body. We then take the ratio between the difference in the elastic energy and the difference in the area of crack. Here are two obvious difficulties:

- In each boundary-value problem, the field is singular around the front of the crack.
- The elastic energies in the two bodies are large quantities compared to their difference. To calculate the energy release rate this way would require us to determine the field very accurately.

Both of the above difficulties are avoided if we use the *J* integral to calculate the energy release rate. The integrand involves the field in the body of a fixed crack. The path of integration needs to enclose the tip of crack, but can be chosen to be far away from the tip of the crack.

References

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