

The Griffith Paper

Readings. A.A. Griffith, The phenomena of rupture and flow in solids. Philosophical Transactions of the Royal Society of London, Series A, Volume 221 (1921) 163-198. This paper initiated the theory of fracture, and has foreshadowed much of the subsequent development. Start to read the paper now, and return to it for illumination later in your career.

Qualitative content of the Griffith theory. After the atomistic nature of matter was confirmed by many experimental observations, about a century ago, it became useful to relate macroscopic phenomena to atomic processes. In 1921 the British engineer A.A. Griffith published a paper on one such macroscopic phenomenon: fracture of a glass. The main puzzle had been that the glass usually breaks under a stress several orders of magnitude below the strength of atomic cohesion.

Griffith took up the notion that a piece of glass is never perfect: small cracks pre-exist in a body of the glass. The tip of such a crack greatly concentrates stress. The intense stress at the tip of the crack breaks atomic bonds one by one, like opening a zipper. The crack advances, leading to the fracture of the body.

This picture is easy to understand, but difficult to quantify. The essential difficulties are

1. The atomic behavior at the tip of the crack is nonlinear, and differs in details for different materials.
2. The shape of the tip is unknown, but the shape of the tip greatly affects the magnitude of stress, at least within the theory of linear elasticity.

It remains unclear what permitted Inglis (1913) to characterize the tip of a flaw by the radius of curvature, and then use a linear elastic solution. A direct approach to resolve these difficulties would be to use a nonlinear material model, and to use more realistic geometry. For example, one may invoke atomistic simulations. Atomistic simulations of fracture have been pursued since the time of Inglis, but have not been widely used in engineering practice to this day.

Griffith took a less direct approach. Consider a pre-existing crack in a body subject to an external force. Regard the body and the external force together as a thermodynamic system, characterized by two thermodynamic variables: the area of the crack and the displacement of the loading grips. To focus on essential ideas, suppose that, after a certain amount of displacement, the loading grips are held fixed, but the crack is allowed to advance. Because the loading grips are held fixed, the external force does no work. The energy of the system is the sum of the elastic energy in the body, and the surface energy in the faces of the crack. The energy of the system is a function of a single thermodynamic variable: the area of the crack. When the crack advances, the stress in the sample is partially relieved, so that the elastic energy is reduced. At the same time, the advancing crack creates more surface area, so that the surface energy increases. Thermodynamics dictates that the process should go in the direction that reduces the total free energy. If the decrease in elastic energy prevails, the crack grows. If the increase in surface energy prevails, the crack heals.

The nonlinear zone, localized around the tip of the crack, remains invariant as the crack advances. Consequently, the presence of the nonlinearity does not affect the accounting of energy. The Griffith approach circumvents the nonlinear crack-tip behavior by invoking one quantity: the surface energy. The science of fracture was born.

To sum up, the qualitative content of the Griffith theory is

- In a body of a glass cracks pre-exist.
- The tip of such a crack concentrates stress.
- The intense stress breaks atomic bonds one by one, like opening a zipper.
- As the crack advances, fresh surfaces are created. The surface energy increases, but the elastic energy decreases.
- The crack advances if the advance reduces the sum of the surface energy and elastic energy.

I'll do two things in this lecture. In formulating his theory, Griffith used this quantity surface energy. For some of us, the last time we encountered the surface energy was in kindergarten, when we blew soap bubbles. So I'll first outline aspects of the surface energy. I'll then describe the content of the Griffith paper.

Surface energy. An atom at the surface of a body has a bonding environment different from that of an atom inside the body, so that the free energy per atom at the surface is higher than that in the body. The excess defines the surface energy.

The above description appeals to intuition, but is not operational. An atom does not have its private energy. An operational definition of the surface energy goes like this. Imagine a body whose size is much larger than an individual atom. Denote the energy of this body by U_0 . It includes all the energy stored in electron clouds, or even the nuclear energy stored in the atomic nuclei. Next imagine that this body is split in two halves. The act of splitting may cause some damage of the material, e.g., introducing some dislocations or some microcracks. Let's say we are very careful in the act of splitting, and anneal the samples afterwards, so that such damage is eliminated. All atoms on the two surfaces relax to their equilibrium configurations. The energy in the two halves will be greater than U_0 . Let's call the energy in the two halves U . The difference, $U - U_0$, is defined as the surface energy. Of course, this difference does not belong to a single layer of atoms on the surface. It is a collective effect. For this definition to be useful, we take advantage of a physical fact: atoms a few layer beneath the surface recover the configuration of atoms in the bulk.

Let γ be the surface energy per unit area. The quantity γ is called the surface energy density, or surface energy for brevity. The surface energy density is independent of the size of the body, unless the body approaches the atomic dimension.

Water: 0.08 J/m². Most other liquids have smaller values.

Metals and ceramics: ~1 J/m²

Polymers: 0.01-0.1 J/m²

- A.W. Adamson, *Physical Chemistry of Surfaces*. Wiley, New York (1990). Chapter 1 describes many phenomena concerning the effect of surface energy for liquids.
- P.G. de Gennes, F. Brochard-Wyart, and D. Quere, *Capillarity and Wetting Phenomena*, Springer (2004).

Measure the surface energy of a liquid. A U-shaped rigid frame is fixed in space. A liquid membrane lies in the area confined by the frame and a slider. The slider can move without friction. The thickness of the membrane is much larger than the dimension of the individual molecule of the liquid. As the slider moves to the right, the membrane becomes thinner and has a larger area. Molecules are drawn from the interior of the membrane to the surfaces. A force F is applied on the slider. What is the force needed to maintain equilibrium? When the slider moves by a distance δx , the area of the membrane increases by $2b\delta x$, and the surface energy increases by $2\gamma b\delta x$. You can always represent a constant force by a weight. When the slider moves by a distance δx , the weight drops by the same distance, so that the gravitational energy decreases by $F\delta x$. The net energy change is the combination of the two effects:

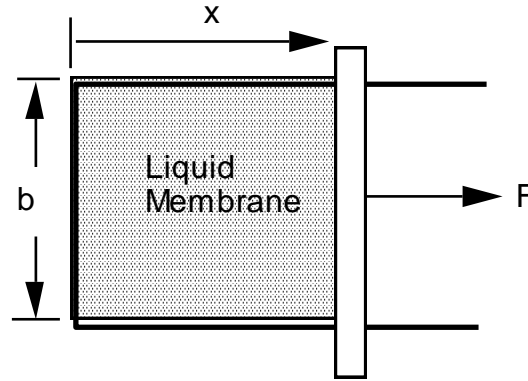
$$\delta U = 2\gamma b\delta x - F\delta x.$$

The two terms compete. The surface energy decreases when the slider moves to the left. On the other hand, the gravitational energy decreases when the slider moves to the right.

The system reaches a state of equilibrium when the variation in the energy associated with the variation in the position of the slider vanishes, namely, when

$$F = 2b\gamma.$$

By measuring the weight F that equilibrates the membrane, one determines the surface energy density. The surface energy density behaves like a force per unit length. Perhaps for this reason the quantity γ is also known as the surface tension.



A liquid droplet on a ceiling: a competition between surface energy and gravity. Let a be the radius of a water droplet on a ceiling. The gravity tends to pull the droplet down, but the surface tension tends to keep it up. In this case, the surface tension of the surface molecules acts like a rubber bag, holding the water inside. Determine the maximum droplet size.

Force balance approach.

$$\gamma a \approx \rho g a^3.$$

The competition between the gravity and the surface energy defines a length scale:

$$a = \sqrt{\frac{\gamma}{\rho g}}$$

Estimation of the maximum size of a water droplet on the ceiling. Take $\gamma = 0.1 \text{ J/m}^2$, $\rho = 1000 \text{ kg/m}^3$, and $g = 10 \text{ m/s}^2$. We find that a is of magnitude of cm.

Measuring the Surface Energy for Solids

Experimental determination of surface energy for solids is challenging. To measure the surface energy, one has to observe a phenomenon in which new surface area is created. For a liquid, the new surface area is created by flow, drawing molecules from the interior of the liquid to the surface. For a solid, one has to do essentially the same thing. Heat the solid up so that it creeps (i.e., flows slowly). In the following words, Griffith described how he measured the surface energy of glass.

Between 730° C and 900° C the method described below was found to be practicable. Fibers of glass about 2 inches long and from 0.002-inch to 0.01-inch diameter, with enlarged spherical ends, were prepared. These were supported horizontally in stout wire hooks and suitable weights were hung on their mid-points. The enlarged ends prevented any sagging except that due to extension of the fibers. The whole was placed in an electric resistance furnace maintained at the desired temperature. Under these conditions viscous stretching of the fiber occurred until the suspended weight was just balanced by the vertical components of the tension in the fiber. The latter was entirely due, in the steady state, to the surface tension of the glass, whose value could therefore

be calculated from the observed sag of the fiber. In the experiments the angle of sag was observed through a window in the furnace by means of a telescope with a rotating cross wire. If w is the suspended weight, d the diameter of the fiber, T the surface tension, and θ the angle at the point of suspension between the two halves of the fiber, then, evidently,

$$\pi d T \sin \frac{\theta}{2} = w.$$

The Griffith theory. A large sheet of a glass is under stress σ . The sheet has unit thickness. For the time being, assume that the loading grips are rigidly held, so that the displacement is fixed, and the loading device does no additional work after a fixed displacement is applied. The state of reference is a stressed sheet with no crack. The state that interests us is the sheet with a crack of length $2a$. We now calculate the difference in energy between the two states.

The surface energy increases by $4a\gamma$.

The elastic energy reduces. To determine the amount of the reduction, one has to solve the boundary-value problem. This tough elasticity problem is for professional elasticians. Look how complicated the stress field must be near the crack. Griffith used the elasticity solution of Inglis, because a crack is just a special case of an ellipse when $b/a \rightarrow 0$. This part of the Griffith paper is difficult to read, and is not very interesting. In the end he made small errors. An alternative approach is to invoke linearity and dimensional considerations. For a linearly elastic problem, the stress field is linearly proportional to the applied stress. The elastic energy per unit volume is proportional to σ^2/E . The elastic energy in an infinite sheet is infinite. However, we are interested in the difference in elastic energy between the cracked sheet and the uncracked sheet. Note that the crack length a is the only length scale in the boundary-value problem. Consequently, the difference in elastic energy between the two sheets takes the form

$$\beta \frac{\sigma^2}{E} a^2,$$

where β is a numerical value. Thus, from very basic considerations, we get nearly everything except for a pure number. This number must be determined by solving the elasticity boundary value problem. The solution turns out to be $\beta = \pi$. You can find the solution of the full problem in Timoshenko and Goodier.

Relative to the uncracked sheet, in the cracked sheet the combined surface energy and elastic energy is

$$U(a) = 4\gamma a - \pi \frac{\sigma^2}{E} a^2.$$

The crack length, $2a$, is the thermodynamic variable. The surface energy density γ and the applied stress σ are taken to be constant for the time being. As expected, when the crack length increases, the surface energy increases, but the elastic energy decreases.

Critical crack size. Plot the free energy as a function of the crack length. The free energy first goes up, reaches a peak, and then goes down. Because there is no minimum free energy, the crack cannot reach equilibrium. The free energy reaches the peak at the crack length

$$a^* = \frac{2\gamma E}{\pi \sigma^2}.$$

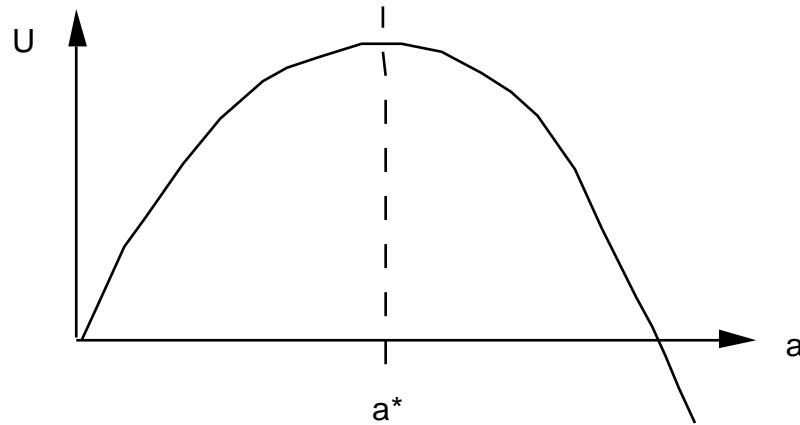
Let us examine the physical significance of this particular crack length. Let a be the length of the crack pre-existing in the sheet. Distinguish two situations.

Crack healing. If $a < a^*$, the surface energy prevails over the elastic energy. To reduce the free energy, the crack length must decrease. The crack does so by healing, i.e.,

forming atomic bonds one by one, like closing a zipper. In reality crack healing is not often observed. This is not because the thermodynamics is wrong, but because surfaces are not flat to the atomic dimension, so that atoms cannot meet across the gap and form bonds. Several examples show that crack healing happens.

- *Adhesives*. Soft material can heal readily by flows.
- *Wafer bonding*. If the surfaces are indeed made flat, they will join.
- *Sintering*. At elevated temperatures, atoms can diffuse, so that the two surfaces change shape and can join.

Crack growth. If $a > a^*$, the elastic energy prevails over the surface energy. To reduce the free energy, the crack must grow. The crack does so by breaking atomic bonds one by one, like opening a zipper. This is the situation studied in this course.



	Crack Length, $2a$ mm	Measured Strength, σ_c MPa	$\sigma_c \sqrt{a}$ MPa $\sqrt{\text{m}}$
sample 1	3.8	6.0	0.26
sample 2	6.9	4.3	0.25
sample 3	13.7	3.3	0.27
sample 4	22.6	2.5	0.27

(Data from the Griffith experiment)

The Griffith experiments. The main prediction of the Griffith theory can be written as

$$\sigma_c \sqrt{a} = \sqrt{\frac{2\gamma E}{\pi}}$$

He performed several experiments to ascertain various parts of the equation.

Experiment 1. Confirm that $\sigma_c^2 a = \text{constan}$, independent of the size of the crack. Start with several glass sheets (large spherical bulbs actually). Introduce a crack in each sheet. Measure the strength of each sheet. Two important points:

- (1) The crack introduced is in the mm to cm range, much longer than any “natural flaws” in the sheets, so that the natural flaws are negligible. In this way Griffith circumvented the uncertainties associated with the natural flaws.

(2) The introduced cracks in different sheets have different lengths, and the measured strengths are also different.

His data confirmed that $\sigma_c^2 a = \text{const}$.

Experiment 2. Confirm that the constant is indeed $\sqrt{2\gamma E/\pi}$. Young's modulus for the glass used by Griffith was $E = 62$ GPa. The surface energy inferred from the measured breaking strength is $\gamma = 1.75$ J/m².

Griffith needed an independent measurement of the surface energy. He did the creeping fiber experiment. The value he obtained was $\gamma = 0.54$ J/m². The agreement was fair.

Experiment 3. Measure strengths of glass fibers.

Experimental strength. For a fixed pre-existing crack size a , there is a critical stress:

$$\sigma_c = \sqrt{\frac{2\gamma E}{\pi a}}.$$

This is the stress needed to fracture the sample. This relation shows that the fracture strength depends on the crack size. Because different samples have different crack sizes, the fracture strength is not a material property. The measured strength has large scatter. Take representative values $\gamma = 1$ J/m², $E = 10^{11}$ N/m², $a = 10^{-6}$ m, the strength is 250 MPa. This corresponds to the experimental strength.

Theoretical strength. Assume that the solid is flawless. There is no stress concentration. The solid breaks when the applied stress is so high to break atomic bonds. If we put $a = 10^{-10}$ m (atomic dimension) into the above formula, we obtain an estimate of the theoretical strength ~ 10 GPa.

Alternatively, a commonly quoted rough estimate of the theoretical strength is

$$S_{th} = \frac{E}{10}.$$

Today the theoretical strength can be calculated by atomistic simulations.

Griffith measured the strength of glass fibers of diameters between 107 μ m and 3.3 μ m. The data scattered, but over trend was that the strength increased as the fiber diameter decreased. He reported a strength of 171 MPa for a fiber of diameter 107 μ m, and a strength of 3.4 GPa for a fiber of diameter 3.3 μ m. The theoretical strength of the glass corresponds to a fiber of the smallest possible (molecular) diameter. He plotted his data, and extrapolated to the molecular diameter, and estimated the theoretical strength to be 12 GPa.