

Finite Deformation: Special Cases

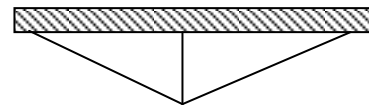
The notes on finite deformation have been divided into two parts: special cases and general theory (<http://imechanica.org/node/538>). In class I start with special cases, and then sketch the general theory. But the two parts can be read in any order. The notes on special cases were initially written for ES 240 Solid Mechanics (<http://imechanica.org/node/203>).

Be wise, linearize. Following this advice of George Carrier, so far in this course we have stayed linear. We have been mostly looking at infinitesimal deformation and Hookean materials. We have mixed the three ingredients: deformation geometry, force balance, and material model. The resulting theory is linear. We have learned fascinating and useful phenomena: stress concentration around a hole, vibration of a beam, refraction of a wave, etc. We have also learned to use commercial finite element code to analyze phenomena with great complexity. We can go on with this linear theory and do a lot more.

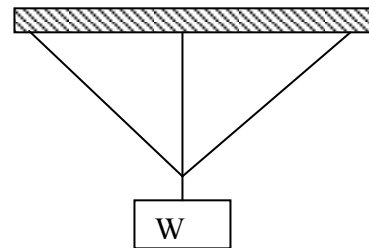
We now wish to go nonlinear, hopefully also with wisdom. We will refine the three ingredients by considering finite deformation and non-Hookean materials. We can mix the refined ingredients, or mix a refined ingredient with unrefined ones. For example, as we have already done, a viscoelastic material is non-Hookean, but deformation of such a material can be infinitesimal. We will outline basic ideas of finite deformation, and describe phenomena that show how finite deformation makes difference.

Finite deformation. When a structure deforms, each state of deformation must obey Newton's law. This principle has often been violated in our past work. For example, in analyzing a truss, we have balanced forces in the deformed truss using the shape of the undeformed truss, neglecting the deformation.

This negligence is often justified on the ground that deformation in most engineering structures is small. You might think that a structure suffering a small strain, say less than 1%, entitles you to neglect the change in shape when you balance forces. A counter example, however, is familiar to you. When a column buckles, the strain in the column is indeed small, but you must enforce equilibrium in the deflected state of the



reference state



current state

column. The essential point is this:

We must enforce Newton's law in every deformed state of a structure, and use this correct principle as a basis to justify any simplification.

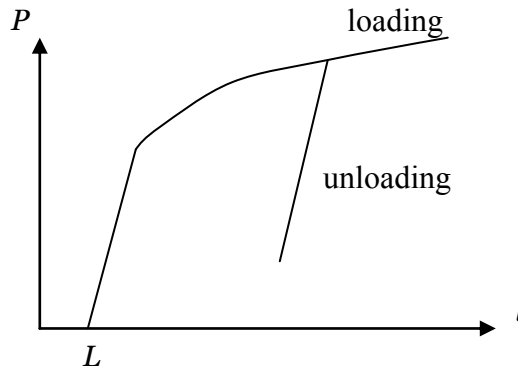
This consideration alone requires us to examine finite deformation, even when the strain is small everywhere in the structure.

Non-Hookean behavior of materials. Hooke's law says that the displacement of a structure is linear in the applied force. This law is an idealization, and contradicts with many daily experiences. Pursuing non-Hookean behavior will take us in many directions. A structure may undergo finite, history-dependent deformation in response to diverse stimuli. Here are some examples.

Nonlinear elasticity. When a force causes a material to deform by a large amount, the displacement may be a nonlinear function of the force. We will talk more about nonlinear elasticity in this set of notes.

Viscoelasticity. We have already looked at time-dependent material behavior, such as viscoelasticity.

Plasticity. After elastic deformation, upon unloading, a metal recovers its shape. After plastic deformation, upon unloading, the metal does not fully recover its shape. Say we apply an axial force to a metal bar, and measure its length. The experimental record of the force-length relation is linear for elastic deformation, and is nonlinear for plastic deformation. During unloading, the metal bar deforms elastically. After plastic loading and elastic unloading, the force-length relation is not one-to-one, but is **history-dependent**. The theory of plasticity will be taught in a separate course.



Deformation in response to diverse stimuli. A material may deform in response to stimuli other than a mechanical force. For example, we have already analyzed deformation caused by change in temperature. Indeed, a material may deform in response to electric field, moisture, light, ionic concentration, etc. We

will pick up this fascinating subject in a subsequent course, ES 241, Advanced Elasticity (<http://imechanica.org/node/725>).

Why do we need stress and strain? We proceed with our subject incrementally, beginning with the simplest structure: a bar. When the bar is unstressed, the cross-sectional area is A and the length is L . We will call this state the reference state. The bar is then subject to an axial force P , and deforms to cross-sectional area a and length l . We will call this state the current state. The experimentalist records, among other things, the force as a function of the length.

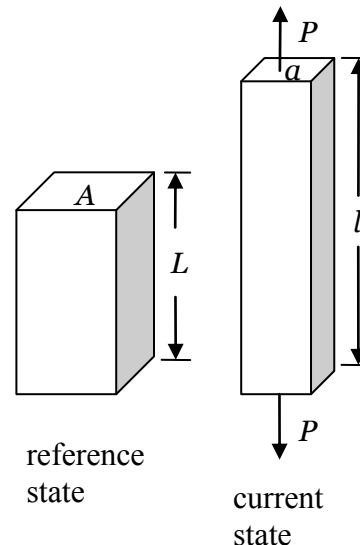
The force-length curve gives us some idea of the behavior of the material. However, we cannot use the curve directly to predict the behavior of another structure made of the same material. To do so, we need to construct a material model that is independent of the shape and size of the bar. As a step forward, we divide the elongation by the length of the bar, and divide the force by the cross-sectional area of the bar. This practice is sensible provided the deformation of the bar is homogenous, and the resulting stress-strain relation is independent of the size of the bar.

This practice is economic. In combination with other ingredients, the stress-strain relation can predict the behavior of any structure made of the same material, even when the structure is of a shape other than a bar, and when deformation is inhomogeneous. We can divide the body into many small volumes, and assume that each small volume undergoes homogeneous deformation.

In the following paragraphs, we will first define strain and stress, and describe the stress-strain relations for particular materials. We then analyze several phenomena.

Strain. The initial, unstressed state is taken as the reference state, in which the length of the bar is L . The deformed state is taken as the current state, in which the length of the bar is l .

Define the **engineering strain** by the elongation of the bar in the current state divided by the length of the bar in the reference state:



$$e = \frac{\text{elongation}}{\text{length in the reference state}} = \frac{l - L}{L}.$$

Another type of strain is defined as follows. Deform the material from a current length l by a small amount to $l + \delta l$. Define the increment in the strain, $\delta \varepsilon$, as the increment in the length of the bar divided by the current length of the bar, namely,

$$\delta \varepsilon = \frac{\text{increment in length}}{\text{length in the current state}} = \frac{\delta l}{l}.$$

This equation defines the increment of **natural strain**. Integrating from L to l , we obtain that

$$\varepsilon = \log\left(\frac{l}{L}\right).$$

Yet one more type of strain, the **Lagrange strain**, is defined as

$$\eta = \frac{1}{2} \left[\left(\frac{l}{L} \right)^2 - 1 \right].$$

This definition is hard to motivate in one dimension. But if you take the view that any one-to-one function of l/L is a definition of strain, then no motivation is needed.

Indeed, even the ratio l/L itself has a name: the **stretch**. Define the stretch as the length of the bar in the current state divided by the length of the bar in the reference state:

$$\lambda = \frac{\text{length in current state}}{\text{length in reference state}} = \frac{l}{L}.$$

There seems to be no lack of ingenuity to invent yet another definition of strain. All these definitions contain the same information: the ratio l/L . For example, every one of the definitions above is a function of the stretch:

$$e = \lambda - 1, \quad \varepsilon = \log \lambda, \quad \eta = \frac{\lambda^2 - 1}{2}.$$

Because they are all one-to-one functions, any one definition can be taken to be “basic” and then used to express all the other definitions. For example, we can express all definitions in terms of the engineering strain:

$$\lambda = e + 1, \quad \varepsilon = \log(1 + e), \quad \eta = \frac{1}{2} e^2 + e$$

When deformation is small, namely, $e \ll 1$, the three definitions of strain are approximately equal, $\varepsilon \approx \eta \approx e$.

When we call e the engineering strain, we do not mean that e is quick-

and-dirty, or unscientific, or unnatural. We just wish to name the quantity $(l-L)/L$. Later on, we will describe motivations for some of the definitions, but these motivations are just elaborate ways to express preferences of individual people. The motivations, however elaborate, should not obscure a simple fact: you can use any one-to-one function of l/L to define strain.

Stress. When dealing with finite deformation, we must be specific about the area used in defining the stress. Define the **nominal stress**, s , as the force applied to the bar in the current state divided by the cross-sectional area of the bar in the reference state:

$$s = \frac{\text{force in the current state}}{\text{area in the reference state}} = \frac{P}{A}.$$

The nominal stress is also known as the engineering stress, or the first Piola-Kirchhoff stress.

Define the **true stress**, σ , as the force in the current state divided by the area in the current state, namely,

$$\sigma = \frac{\text{force in the current state}}{\text{area in the current state}} = \frac{P}{a}.$$

The true stress is also known as the Cauchy stress

Once again, you should not be misled by the names, or influenced by the prejudice of your teachers. The true stress is no truer than the nominal stress. The engineering stress is no less scientific. They are just different definitions of stress, and we need to have different names for them.

Work-conjugate strain and stress. It helps if our definitions of stress and strain ease the calculation of work. When the bar elongates from length l to length $l + \delta l$, the force P does work $P\delta l$. Recall one pair of definitions of stress and strain:

$$P = sA, \quad l = \lambda L.$$

Consequently the work done by the force is

$$P\delta l = ALs\delta\lambda.$$

Since AL is the volume of the bar in the reference state, we note that

$$s\delta\lambda = \frac{\text{increment of work in the current state}}{\text{volume in the reference state}}.$$

We say that the nominal stress and the stretch are *work-conjugate*. Also note that $\delta\varepsilon = \delta\lambda$, so that the nominal stress is also work-conjugate to the engineering strain.

Recall another pair of definitions of stress and strain:

$$P = \sigma A, \quad d = l \delta \varepsilon.$$

The work done by the force is

$$P d = A l \sigma \delta \varepsilon.$$

Since $A l$ is the current volume of the bar, we note that

$$\sigma \delta \varepsilon = \frac{\text{increment of work in the current state}}{\text{volume in the current state}}.$$

That is, the true stress is work-conjugate to the natural strain.

Given any definition of strain, we can define its work-conjugate stress. For example, consider the Lagrange strain, η . Subject to an increment in the strain, $\delta \eta$, the force acting on the element does the work. Denote

$$S \delta \eta = \frac{\text{increment of work in the current state}}{\text{volume in the reference state}}.$$

This expression gives a new definition of stress, S . This new definition does not have a “simpler” interpretation than its status as the work conjugate to the Lagrange strain.

If we are liberal about the definition of strain, without being obsessive about “motivating” each definition, we may as well take a liberal view to call the work conjugate of each definition of strain a definition of stress, and name our definition after a mechanician who can no longer protest.

We can easily invent and name definitions, but all the above definitions have already had names:

- σ : the true stress or the Cauchy stress.
- s : the nominal stress or the first Piola-Kirchhoff stress.
- S : the second Piola-Kirchhoff stress.

Recall the relations among the definitions of strain:

$$e = \lambda - 1, \quad \varepsilon = \log \lambda, \quad \eta = \frac{\lambda^2 - 1}{2}$$

We obtain the relations among their increments:

$$\delta e = \delta \lambda, \quad \delta \varepsilon = \frac{\delta \lambda}{\lambda}, \quad \delta \eta = \lambda \delta \lambda.$$

Consequently, the three definitions of stress are related as

$$\sigma A = s A, \quad s = \lambda S.$$

Are these alternative definitions of strain and stress necessary?

I have my opinion, and you may have yours. The question perhaps boils down to something no more profound than asking, “Is it necessary to know many alternative roads to Boston Common?” Whatever your opinions are, however, it

may alleviate some of your agony in studying the subject by knowing that textbooks of nonlinear continuum mechanics are full of alternatives at every turn. These alternatives hide behind sophisticated symbols and seductive motivations. You will just have to look beyond them, and focus on matters of substance.

Material models. Using a bar of given material, our experiment records a force-length curve. We then convert the curve to a stress-strain relation by using some definitions of stress and strain. Any definitions serve the purpose. All that matters is that we tell other people which definitions we have used.

For a **metal** undergoing large, plastic deformation, the stress-strain curve (without unloading) is often fit to a power law in terms of the true stress and the natural strain:

$$\sigma = K\varepsilon^N,$$

where K and N are parameters to fit experimental data. Some representative values: $N = 0.15-0.25$ for aluminum, $N = 0.3-0.35$ for copper, $N = 0.45-0.55$ for stainless steel. K has the dimension of stress; it represents the true stress at strain $\varepsilon = 1$. Representative values for K are 100 MPa – 1GPa.

For a **rubber**, the stress-strain data may be fit to a relation known as the neo-Hookean law:

$$s = \mu(\lambda - \lambda^{-2}).$$

Representative values for μ are 0.1 MPa – 10 MPa.

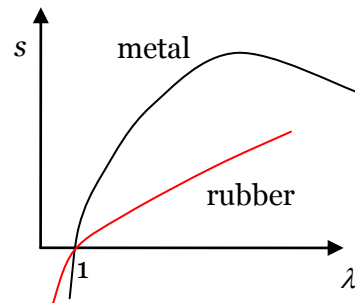
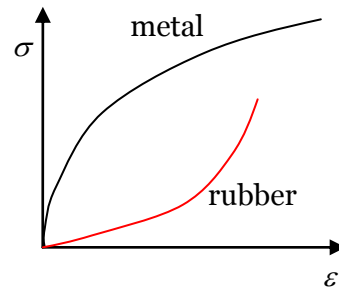
When a material undergoes large deformation, volumetric strain is often negligible compared to tensile strain. Consequently, such a material may be taken to be incompressible.

To see the difference in material behavior, we should use the same definition of stress and strain for both materials. Recall that $\sigma = s\lambda$ (for incompressible materials) and $\varepsilon = \log \lambda$. In terms of the nominal stress and the stretch, the two material models are

$$s = K \frac{(\log \lambda)^N}{\lambda}, \quad \text{for metals}$$

$$s = \mu(\lambda - \lambda^{-2}), \quad \text{for rubbers}$$

In terms of the true stress and the natural



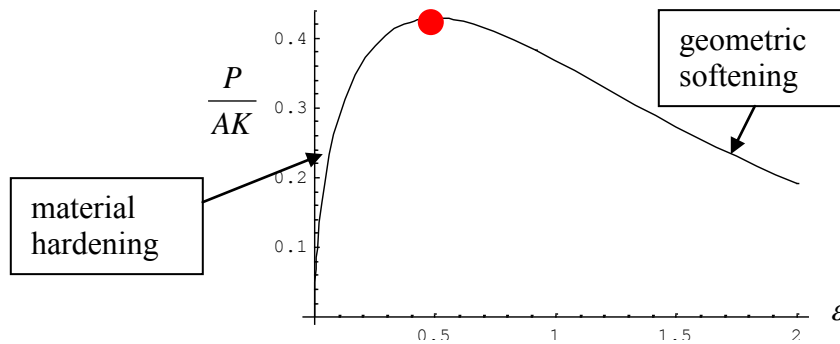
strain, the two material models are

$$\sigma = K\varepsilon^N, \quad \text{for metals}$$

$$\sigma = \mu[\exp(2\varepsilon) - \exp(-\varepsilon)], \quad \text{for rubbers}$$

This change of variables makes evident a key difference: in tension, the stress in rubbers rises more steeply than in metals. We will return to this difference shortly.

Exercise. Use the 3 ingredients outlined about to obtain the force-deflection relation for the truss sketched in the beginning of the notes. Assuming that all three members of the truss are made of rubber bands, and that deformation is large.



Necking in a metal bar. Let us mix the newly refined three ingredients to analyze a specific phenomenon: necking. Subject to a tensile force, a metal bar first elongates to a homogenous state of strain. At some level of strain, a small part of the bar thins down preferentially, forming a neck. By contrast, a rubber band under tension usually deforms homogeneously, and does not form a neck. We would like to understand these observations.

Here is a summary of the three ingredients:

- Deformation geometry: $\varepsilon = \log(l/L)$.
- Force balance: $P = \sigma a$.
- Material model: For a metal bar under uniaxial tension, the true stress relates to the natural strain as $\sigma = \sigma(\varepsilon)$. We will assume that the volume of the bar is constant during deformation, $AL = al$, or $a = A \exp(-\varepsilon)$.

Mixing the three ingredients, we obtain the force as a function of strain:

$$P = A\sigma(\varepsilon)\exp(-\varepsilon).$$

For metals, we use material model $\sigma = K\varepsilon^N$ and obtain that

$$P = AK\varepsilon^N \exp(-\varepsilon).$$

Plot P as a function of ε . In plotting the figure, I've set $N = 0.5$. Observe the two competing factors: material hardening and geometric softening. As the bar elongates, the material hardens, as reflected by the hardening exponent in the stress-strain relation $\sigma = \sigma(\varepsilon)$. At the same time, the elongation reduces the cross-sectional area, an effect known as geometric softening. For small deformation, $P \rightarrow 0$ as $\varepsilon \rightarrow 0$; material hardening prevails, and the force increases as the bar elongates. For large deformation, so long as the stress-strain relation $\sigma(\varepsilon)$ increases slower than $\exp(\varepsilon)$, $P \rightarrow 0$ as $\varepsilon \rightarrow \infty$, geometric softening prevails, and the force drops as the bar elongates.

To determine the peak force, note that

$$\frac{dP}{d\varepsilon} = A \exp(-\varepsilon) \left[\frac{d\sigma(\varepsilon)}{d\varepsilon} - \sigma \right].$$

Consequently, the force P peaks when the true stress equals the tangent modulus:

$$\frac{d\sigma(\varepsilon)}{d\varepsilon} = \sigma(\varepsilon).$$

This equation, known as the **Considère condition**, determines the strain at which the force peaks. For the power-law material, $\sigma(\varepsilon) = K\varepsilon^N$, the force peaks at the strain

$$\varepsilon_c = N.$$

When the metal bar is pulled beyond the critical strain, deformation becomes inhomogeneous, with a segment of the bar elongates at a higher strain than the rest of the bar. That is, a neck forms in the bar. After a neck forms, the field of stress in the bar is no longer uniaxial, but is triaxial. Such an inhomogeneous deformation can be analyzed by using the finite element method. For example, the necking process can be studied by using ABAQUS.

- A. Needleman, A numerical study of necking in circular cylindrical bar, *Journal of the Mechanics and Physics of Solids*, 20, 111 (1972).

A rubber band under tension does not form a neck. For a rubber band, assume the material is Neo-Hookean:

$$\sigma = \mu(\lambda^2 - \lambda^{-1}) = \mu[\exp(2\varepsilon) - \exp(-\varepsilon)].$$

Thus, at a large tensile strain, the true stress increases exponentially with the natural strain, so that the force $P = A\sigma(\varepsilon)\exp(-\varepsilon)$ always increases with the strain. The rubber band will not form a neck under uniaxial tension.

Exercise. Argue that necking sets in at a stretch where nominal stress is maximum. Thus, in terms of the nominal stress and the stretch, the Considère condition is

$$\frac{ds(\lambda)}{d\lambda} = 0.$$

Apply this condition to metals and rubbers. This exercise shows that the same physical condition may take different forms when different definitions of stress and strain are used. In this case, the use of nominal stress simplifies the discussion.

A metal bar pulled beyond the critical strain. To gain insight into the post-necking behavior, we look at a simplified model: the long-neck model. The neck is assumed to be long compared to the diameter of the bar, so that the deformation in the neck is homogenous, characterized by the stretch λ_{neck} . The deformation in the remainder of the bar is also homogeneous, characterized by another stretch, λ_0 . The neck stretches more than the rest of the bar:

$$\lambda_{neck} > \lambda_0.$$

To balance the forces, the nominal stress in the neck equals the nominal stress in the rest of the bar:

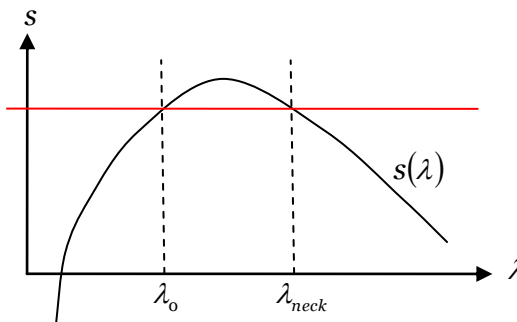
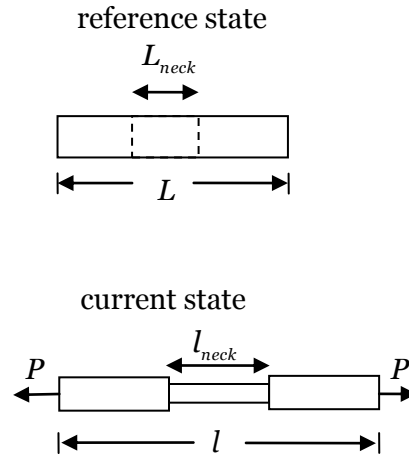
$$s(\lambda_{neck}) = s(\lambda_0).$$

When the stress-stretch curve $s(\lambda)$ is not monotonic, necking is possible.

Let L_{neck} be the length of the neck in the reference state. The length of the bar in the current state is

$$l = L_{neck}\lambda_{neck} + (L - L_{neck})\lambda_0.$$

This idealized model enables us to interpret several experimental observations.



Effect of strain rate on necking. At an elevated temperature, glass flows and can be drawn into a thin fiber without forming necks. The rate-dependent deformation also has significant effect on the stability of metals even at the room temperature. Metals appear to be harder when they are loaded at a higher strain rate. A material model capturing this effect of strain rate is

$$\sigma = K\varepsilon^N \left(\frac{d\varepsilon}{dt} \right)^m,$$

where m is the strain-rate hardening exponent. Experiments indicate that this strain-rate sensitivity can significantly increase the elongation of a metal before fracture. This observation may be understood as follows. Consider an imperfect bar, with a segment thinner than the rest of the bar. When the bar is pulled, the thinner region has a higher strain rate, and is therefore harder, than the rest of the bar. This effect of strain rate counteracts necking.

The axial force is $P = \sigma a$. Due to the assumption of incompressibility, the cross-sectional area relates to the strain as $a = AL/l = A \exp(-\varepsilon)$. Thus, the axial force relates to the strain as

$$P = KA \exp(-\varepsilon) \varepsilon^N \left(\frac{d\varepsilon}{dt} \right)^m.$$

Now apply this expression to the imperfect bar. In the unstressed state, let the cross-sectional area of the thinner segment of the bar be A_{neck} , and let the cross-sectional area of the remaining parts of the bar be A_0 . In the current state, when the bar is subject to the axial force P , everywhere in the bar sustains the same axial force. Denote the strain in the thinner segment of the bar by ε_{neck} , and the strain in the rest parts of the bar by ε_0 . The equality of the axial force in the bar requires that

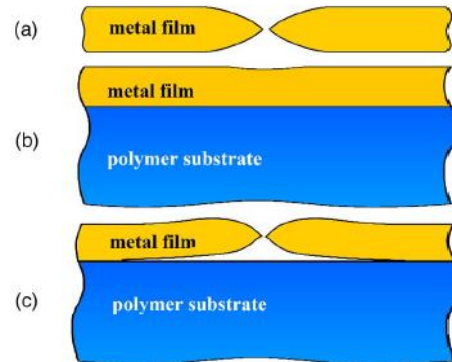
$$\left(\frac{A_{neck}}{A_0} \right)^{\frac{1}{m}} \exp\left(-\frac{\varepsilon_{neck}}{m}\right) \varepsilon_{neck}^{N/m} d\varepsilon_{neck} = \exp\left(-\frac{\varepsilon_0}{m}\right) \varepsilon_0^{N/m} d\varepsilon_0.$$

This equation can be used to plot ε_{neck} against ε_0 . See descriptions of the results and comparison with experiments in the following paper.

- J.W. Hutchinson and K.W. Neale, Influence of strain-rate sensitivity on necking under uniaxial tension, *Acta Metallurgica* 25, 839-846 (1977). <http://www.seas.harvard.edu/hutchinson/papers/340.pdf>

Freestanding thin metal wires and thin metal films fracture at small strains. When a freestanding film of a plastically deformable metal is

subjected to a tensile load, the film ruptures by strain localization, by forming a neck within a narrow region, of a width comparable to the thickness of the film. The strain is large within the neck, but is small elsewhere in the film. Because the film has an extraordinarily large length-to-thickness ratio, the net elongation of the film upon rupture is small, typically less than a few percent.



Let L_{neck} be the length of the neck in the reference state. The length of the bar in the current state is

$$l = L_{neck}\lambda_{neck} + (L - L_{neck})\lambda_0.$$

The apparent stretch of the bar is

$$\frac{l}{L} = \frac{L_{neck}}{L}\lambda_{neck} + \left(1 - \frac{L_{neck}}{L}\right)\lambda_0.$$

The neck is highly localized: the length of the neck is typically on the order of the thickness of the film,

$$L_{neck} \approx H.$$

For a thin film, the length of the neck is much shorter than the length of the bar, $L_{neck}/L \ll 1$. Consequently, the apparent stretch of the bar is $l/L \approx \lambda_0$.

A thin metal film bonded to a polyimide substrate can be stretched beyond 50%. For a metal film well bonded to a polymer substrate, the substrate can delocalize strain, so that the metal film can elongate indefinitely, only limited by the rupture of the polymer substrate. Rupture of a thin metal film by strain localization: (a) Freestanding film: local elongation leads to rupture; (b) Supported film: local elongation suppressed by substrate; (c) Debonding assists in rupture.

- N.S. Lu, X. Wang, Z.G. Suo and J. J. Vlassak, Metal films on polymer substrates stretched beyond 50%. Applied Physics Letters 91, 221909 (2007). <http://www.seas.harvard.edu/suo/papers/201.pdf>

Inhomogeneous, time-dependent deformation. Consider inhomogeneous and time-dependent deformation of a bar. We next list the ingredients of the theory.

Kinematics. The bar in the undeformed state is taken to be the reference state. The bar is modeled as a field of material particles. Name each material

particle by the coordinate X of its place when the bar is in the reference state. In the current state t , the material particle X moves to a place of coordinate x . The kinematic of the deformation of the bar is described by the time-dependent field

$$x = x(X, t).$$

Evolving this field is the aim of the theory.

Consider a differential element of the bar. In the reference state, the element is between two material particles, X and $X + dX$. In the current state at time t , the material particle X moves to place $x(X, t)$, and the material particle $X + dX$ moves to place $x(X + dX, t)$. Recall that the stretch is defined as

$$\lambda = \frac{\text{length in current state}}{\text{length in reference state}}.$$

Thus, the stretch of the differential element is

$$\lambda = \frac{x(X + dX, t) - x(X, t)}{dX},$$

or

$$\lambda = \frac{\partial x(X, t)}{\partial X}.$$

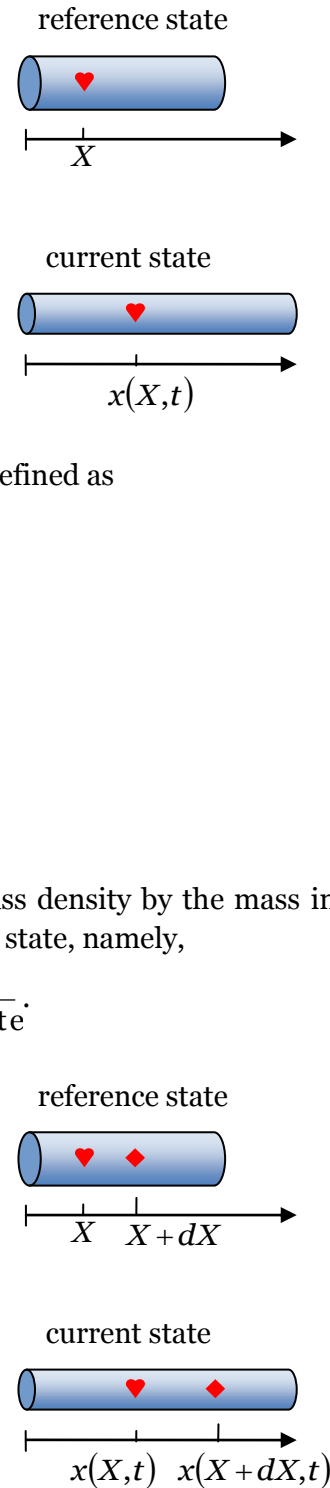
The stretch is also a time-dependent field, $\lambda(X, t)$.

Conservation of mass. Define the nominal mass density by the mass in the current state divided by the volume in the reference state, namely,

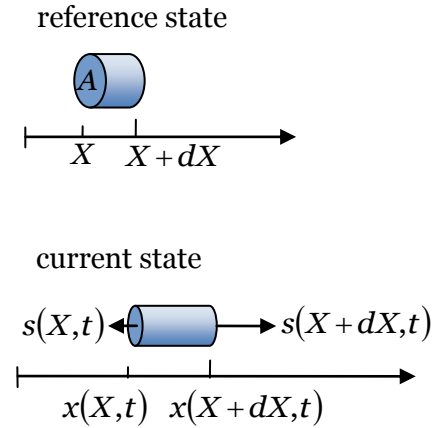
$$\rho = \frac{\text{mass in the current state}}{\text{volume in the reference state}}.$$

We assume that that no mass transfers from one material particle to another. The conservation of mass dictates that the nominal mass density be time-independent. The density, however, may vary from one material particle to another. We may write the nominal mass density as a function of material particles, $\rho(X)$. In the following analysis, we assume that the bar is uniform in the reference state, so that the density is constant independent of time and particle.

Newton's second law. The nominal stress



in the bar also varies from one material particle to another, and from one time to another. Consequently, the nominal stress in the bar is a time-dependent field, $s(X,t)$. When the bar is in the reference state, its cross-sectional area is A . Draw a free-body diagram of a differential element of the bar, between particles X and $X+dX$. The mass of the element is ρAdX . The acceleration of the element is $\partial^2 x(X,t)/\partial t^2$. The net force applied on the element is $[s(X+dX,t)-s(X,t)]A$. Applying Newton's second law to this element at the current state at time t , we obtain that



$$[s(X+dX,t)-s(X,t)]A = (\rho AdX) \frac{\partial^2 x(X,t)}{\partial t^2}.$$

This equation is re-written as

$$\frac{\partial s(X,t)}{\partial X} = \rho \frac{\partial^2 x(X,t)}{\partial t^2}.$$

Material model. The material is taken to be nonlinearly elastic with the stress-stretch relation

$$s = g(\lambda).$$

This function is measured by applying to a short segment of the bar a uniaxial force, which causes a state of homogeneous deformation. We assume that the same function is valid locally for inhomogeneous deformation of the bar.

Summary of the three ingredients. The bar is characterized by three time-dependent fields: deformation $x(X,t)$, stretch $\lambda(X,t)$, and nominal stress $s(X,t)$. The stretch is defined by the gradient of the deformation:

$$\lambda = \frac{\partial x(X,t)}{\partial X}.$$

Newton's law relates the fields of stress and deformation:

$$\frac{\partial s(X,t)}{\partial X} = \rho \frac{\partial^2 x(X,t)}{\partial t^2}.$$

A material model is specified by a stress-stretch relation:

$$s = g(\lambda).$$

The two partial differential equations are linear. The material model is an algebraic relation between the stress and the stretch, which in general is nonlinear. A class of solutions involves linearization of the material model, so that all equations are linear. An example follows.

Wave in a pre-stressed bar. Linear perturbation. We now mix the three ingredients for a specific phenomenon. A bar has a uniform density and a uniform cross-sectional area. The bar is in a state of homogenous deformation of stretch λ_0 . We then launch into the bar a longitudinal wave of small amplitude. What will be the speed of the wave?

This example will illustrate a method known as *linear perturbation*. We begin with a known state of finite deformation. In this example, this state is a bar undergoing a static and homogenous deformation. Superimposed on the homogenous deformation is an inhomogeneous, time-dependent field of displacement $u(X,t)$. In the current state t , the material particle X moves to a place with the coordinate

$$x(X,t) = \lambda_0 X + u(X,t).$$

Taking derivative with respect to X , we obtain the stretch in the bar:

$$\lambda = \lambda_0 + \frac{\partial u(X,t)}{\partial X}.$$

The stretch in the bar is a time-dependent and inhomogeneous field, given by the sum of the homogenous stretch due to the pre-stress and an inhomogeneous strain due to the disturbance.

In the linear-perturbation analysis, the unperturbed state can be a state of finite deformation, but the perturbation in strain is assumed to be small, namely,

$$\left| \frac{\partial u(X,t)}{\partial X} \right| \ll 1.$$

This assumption enables us to use the Taylor expansion.

The material is taken to be nonlinearly elastic, characterized by a function between the stress and the stretch $s = g(\lambda)$. We expand the function $s = f(\lambda)$ into the Taylor series around λ_0 :

$$s(\lambda) = g(\lambda_0) + g'(\lambda_0)(\lambda - \lambda_0).$$

Because the deviation from the homogeneous stretch, $\lambda - \lambda_0$, is small, the expansion is carried to the term linear in the deviation. The first derivative

$$g'(\lambda_0) = \left. \frac{ds(\lambda)}{d\lambda} \right|_{\lambda=\lambda_0}$$

is the tangent stiffness.

A mix of the three ingredients leads to the equation of motion:

$$g'(\lambda_0) \frac{\partial^2 u(X,t)}{\partial X^2} = \rho \frac{\partial^2 u(X,t)}{\partial t^2}.$$

This is a familiar linear partial differential equation, the wave equation. The equation evolves the disturbance $u(X,t)$. Consequently, the disturbance propagates in the bar at the speed

$$c = \sqrt{\frac{g'(\lambda_0)}{\rho}}.$$

The wave slows down when the tangent modulus reduces.

Considère condition reconsidered. The above analysis is sensible only when the tangent modulus is positive, $g'(\lambda_0) > 0$. When the tangent modulus vanishes, $g'(\lambda_0) = 0$, a time-independent inhomogeneous deformation becomes possible. Observe that the condition of vanishing tangent modulus is the same as the Considère condition.

In the previous treatment of necking, we restricted ourselves to homogeneous and time-independent fields. We found that the nominal stress reaches a peak at a certain stretch. We then jumped to the conclusion, asserting that the peak of the $s(\lambda)$ curve corresponds to the onset of necking. That model of necking was unsatisfactory. The model did not show the most salient feature of necking: an inhomogeneous deformation. The model could not continue the calculation after the peak.

Now we have a model of inhomogeneous and time-dependent deformation. Given a boundary condition and an initial condition, the model evolves the field of deformation $x(X,t)$. In particular, we can start with an initial condition of an inhomogeneous deformation. The magnitude of the inhomogeneity can be kept small to represent the imperfection of the bar. We then evolve the field in time, and see if the inhomogeneity amplifies. We can also vary the rate of pulling, and study the effect of inertia.

Of course, we have to be careful about any conclusion we draw from such a model. After all, necking is a three-dimensional process, and our model is one-dimensional. For example, the solution to the one-dimensional model is suspect if inhomogeneity occurs over a length shorter than the diameter of the bar.

Exercise. Small-amplitude oscillation around a state of equilibrium of finite deformation. Formulate the problem. What happens when the frequency of the fundamental mode approaches zero?

A column subject to a compressive axial force. The connection between instability and dynamics also arises in another pair of familiar phenomena: vibration and buckling. Demonstrate in class the effect of axial force on frequency in class using a column with a variable length.

A column, length L , is simply supported at the two ends. When the column has a small deflection w , we balance the moment in the current state:

$$M = Pw.$$

(We will miss this term if we balance the moment in the reference configuration.)

The moment-curvature relation is

$$M = -EI \frac{\partial^2 w}{\partial X^2}.$$

The equilibrium equation becomes

$$EI \frac{\partial^2 w}{\partial X^2} + Pw = 0.$$

The function $w = \sin(\pi X / L)$ satisfies the equilibrium equation and the boundary conditions, provided that the axial force is given by

$$P_c = \pi^2 \frac{EI}{L^2}.$$

This is the Euler condition for buckling.

If the column vibrates in the transverse direction, the equation of motion for deflection $w(X, t)$ is

$$EI \frac{\partial^4 w}{\partial X^4} + P \frac{\partial^2 w}{\partial X^2} = -\rho A \frac{\partial^2 w}{\partial t^2}.$$

Try the solution of the form

$$w(X, t) = \sin\left(\frac{\pi X}{L}\right) \sin \omega t.$$

This form satisfies the boundary conditions of simply supported ends. This form also satisfies the equation of motion, provided the frequency is

$$\omega = \pi^2 \sqrt{\frac{EI}{\rho AL^4} \left(1 - \frac{P}{P_c}\right)}.$$

The frequency decreases when the compressive force increases, and vanishes when the axial force approaches the critical load for buckling.

Inflation of a balloon. When a spherical balloon is in the unstressed state, the radius of the balloon is R , and the thickness of its membrane is H . When the balloon is subject to a pressure p , the radius of the balloon expands to

r , and the thickness of the membrane reduces to h . We would like to relate the radius r and the pressure p .

Kinematics. In the current state, the hoop stretch of the balloon is

$$\lambda = r / R .$$

The material is taken to be incompressible, so that $hr^2 = HR^2$, or

$$h = H\lambda^{-2} .$$

Force balance. The stress normal to the membrane is between p and zero, and is small compared to the hoop stress σ . Consequently, we assume that the membrane is in a state of equal-biaxial stress. Draw the free-body diagram of a half balloon in the current state, and balance the forces: $2\pi r h \sigma = \pi r^2 p$. Thus

$$p = \frac{2h}{r} \sigma .$$

Because $r \gg h$, we confirm that $\sigma \gg p$.

Material model. Suppose that we have tested the material, and obtained the stress-stretch relation in the state of equal-biaxial stress:

$$\sigma = \mu \lambda^{-4} (\lambda^6 - 1) .$$

Mix these ingredients together, and we obtain that

$$p = \frac{2H\mu}{R} \lambda^{-7} (\lambda^6 - 1) .$$

This is the desired relation between the pressure and the stretch. Sketch the function $p(\lambda)$. When the balloon is unstretched, $\lambda = 1$, the pressure is zero, $p(1) = 0$. As the balloon expands, the material stiffens, but the membrane thins. For the neo-Hookean material the above calculation shows that $p(\infty) = 0$. When the stretch is large enough, the geometric thinning prevails over material stiffening, and the pressure reaches a peak. This peak is reached at the critical radius

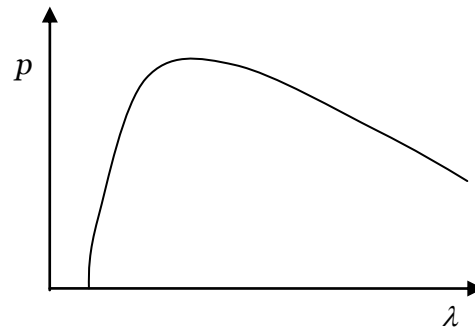
$$\lambda_c = 7^{1/6} \approx 1.38 .$$

The peak pressure is

$$p_c = 1.2\mu H / R .$$

When the pressure approaches this peak value, this model predicts that the balloon will rupture.

The critical radius above seems to be too small compared to our experience with balloons. Perhaps two reasons account for this discrepancy. First, when



we blow a balloon, our lungs cannot supply air at a constant pressure. When the balloon expands significantly, the pressure drops. Second, when the stretch is large, the rubber may stiffen significantly, at a rate much higher than that described the neo-Hookean model.

- I. Muller and H. Struchtrup, Inflating a rubber balloon, *Mathematics and materials of Solids* 7, 569-577 (2002).
- H. Alexander, Tensile instability of initially spherical balloons. *Int. J. Engng. Sci* 9, 151-160 (1971).
- A. Needleman, Inflation of spherical rubber balloons, *International Journal of Solids and Structures* 13, 409-421 (1977).
- D.K. Bogen and T.A. McMahon, Do cardiac aneurysms blow out? *Biophysics Journal* 27, 301-316 (1979).

Exercise. When the balloon is subject to a constant pressure, allow the balloon to oscillate by changing its radius with time. Determine the natural frequency. When does the natural frequency vanish? Link this condition to the condition of instability.

Cavitation. Consider small a cavity in a large block of a material. Remote from the cavity, the material is in a state of hydrostatic tension S . For a brittle material, the cavity concentrates stress, so that a crack may emanate from the cavity. In such a case, deformation is small when the stress reaches a critical value, so that we can analyze the problem assuming that the material is linearly elastic and deformation is infinitesimal. Formulated this way, the problem is known as the Lamé problem, as we have seen before. The key result of the analysis is that the hoop stress at the surface of cavity is $3/2$ times the remote stress.

For a material capable of large deformation, such as a ductile metal or a rubber, however, the cavity may cause another mode of failure. Under the hydrostatic tension remote from the cavity, the cavity may expand indefinitely when the applied stress reaches some finite value, a phenomenon known as cavitation. To study this phenomenon, we need to determine the radius of the cavity as a function of the remote tension. The Lamé solution clearly gives an erroneous prediction in this case because it says that the radius of cavity increases linearly with the applied stress. We next analyze the phenomenon allowing finite deformation. The three ingredients of solid mechanics take the following form.

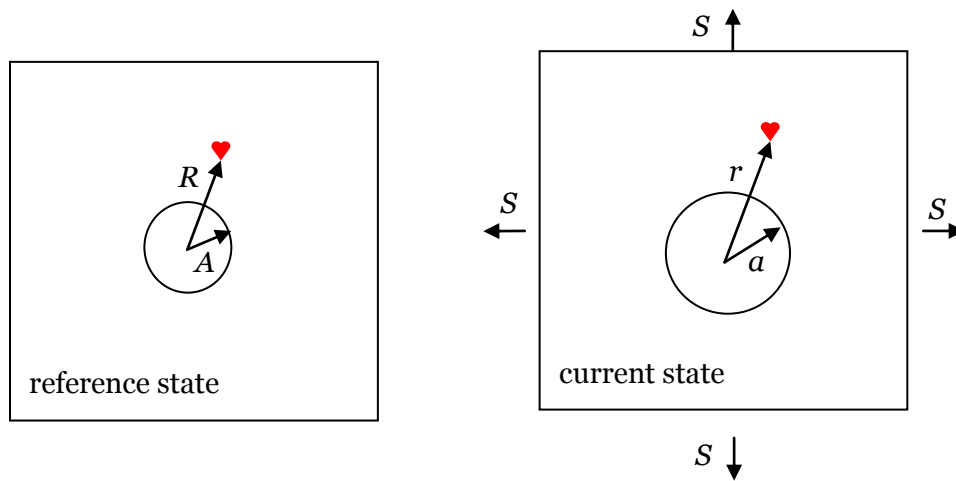
Kinematics. In the reference state, the body is unstressed, the radius of the cavity is A , and the distance between a material particle and the center of the

cavity is R . In the current state, the body remote from the cavity is subject to a triaxial tensile stress S , the radius of the cavity becomes a , and the material particle R moves to a place a distance r from the center of the cavity. The function $r(R)$ characterizes the kinematics. The hoop stretch is

$$\lambda_\theta = \frac{r}{R}.$$

The radial stretch is

$$\lambda_r = \frac{dr(R)}{dR}.$$



Balance of Forces. In the current state, the state of stress in the body remote from the cavity is equal-triaxial, but the state of stress in the body near the cavity is equal-biaxial. That is, the state of stress in the body is inhomogeneous. A material particle at location r is subject to a state of triaxial stress: $\sigma_r(r)$, $\sigma_\theta(r) = \sigma_\phi(r)$. We balance force for a material particle in the current state, so that

$$\frac{d\sigma_r(r)}{dr} + 2\frac{\sigma_r - \sigma_\theta}{r} = 0.$$

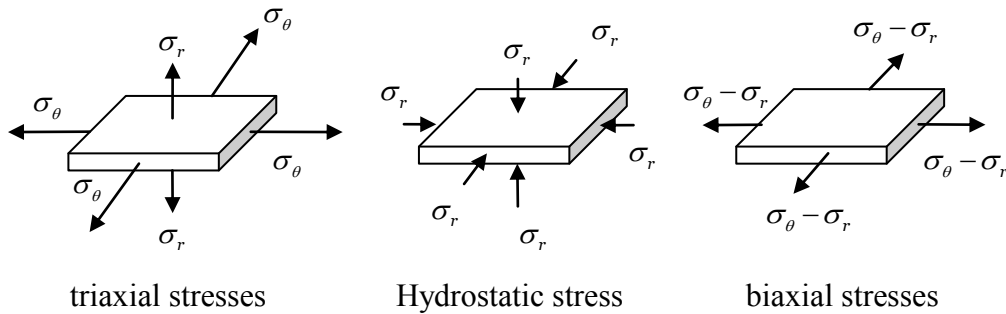
Material model. We assume that the material is incompressible. This assumption markedly simplifies the problem. In the reference state, the volume of the shell between radii A and R is $(4\pi/3)(R^3 - A^3)$. In the current state, the shell deforms and the radii become a and r , and the volume of the shell is $(4\pi/3)(r^3 - a^3)$. Because the material is assumed to be incompressible, the volume of the shell in the reference state equals that in the current state:

$$r^3 - a^3 = R^3 - A^3.$$

Consequently, once we determine the radius of the cavity in the current state, a , we will know the entire field of deformation $r(R)$. The assumption of incompressibility reduces the deformation of the body to a single degree of freedom: the radius of the cavity a .

Suppose that we have tested the material under the equal-biaxial state of stress. When a sample of the material is stretched by equal-biaxial state of stress, $(\sigma, \sigma, 0)$, the material deforms into a state of stretch $(\lambda, \lambda, \lambda^{-2})$. Because the material is taken to be incompressible, the state of stretch under equal-biaxial stress is characterized by a single parameter, λ . Let us say that the experimental measurement gives us a stress-stretch curve in the state of equal-biaxial stress, written as

$$\sigma = g(\lambda).$$



A material particle is under a triaxial state of stress: one radial component and two hoop components, $(\sigma_r, \sigma_\theta, \sigma_\theta)$. The material particle is also in a triaxial state of stretch, $(\lambda_r, \lambda_\theta, \lambda_\theta)$. Because the material is taken to be incompressible, the radial stretch relates to the hoop stretch as $\lambda_r = \lambda_\theta^{-2}$. Also because the material is taken to be incompressible, superposing a hydrostatic stress on the material particle will not change the state of deformation of the particle. For example, we can superimpose $(-\sigma_r, -\sigma_r, -\sigma_r)$ on the material particle, so that the stress state of the particle becomes $(0, \sigma_\theta - \sigma_r, \sigma_\theta - \sigma_r)$. This is an equal-biaxial state of stress. Thus,

$$\sigma_\theta - \sigma_r = g(\lambda_\theta).$$

Because of the spherical symmetry of this phenomenon, we will only need the stress-strain curve determined in the equal-biaxial state of stress to analyze the process of cavitation.

We now mix the three ingredients. Integrating the force-balancing

equation $d\sigma_r/dr = 2(\sigma_\theta - \sigma_r)/r$, we obtain that

$$S = 2 \int_a^\infty \frac{g(\lambda_\theta) dr}{r}.$$

We have used the boundary conditions: $\sigma_r(a) = 0$ and $\sigma_r(\infty) = S$. Recall that $\lambda_\theta = r/R$, and that the elastomer is taken to be incompressible, $r^3 - a^3 = R^3 - A^3$. Consequently, r is a function of λ_θ , namely,

$$r = (a^3 - A^3)^{1/3} \lambda_\theta (\lambda_\theta^3 - 1)^{-1/3}.$$

In the integral above, we change the variable from r to λ_θ , and we obtain that

$$S = -2 \int_1^{a/A} \frac{g(\lambda_\theta) d\lambda_\theta}{\lambda_\theta (\lambda_\theta^3 - 1)}.$$

Once the stress-stretch curve $g(\lambda)$ is prescribed, this equation relates the remotely applied stress, S , to the stretch of the cavity, a/A .

For the Neo-Hookean material, the stress-stretch relation in the state of equal-biaxial stress is

$$g(\lambda) = \mu \lambda^{-4} (\lambda^6 - 1).$$

The integral can be evaluated analytically, giving

$$\frac{S}{\mu} = \frac{5}{2} - 2 \left(\frac{a}{A} \right)^{-1} - \frac{1}{2} \left(\frac{a}{A} \right)^{-4}.$$

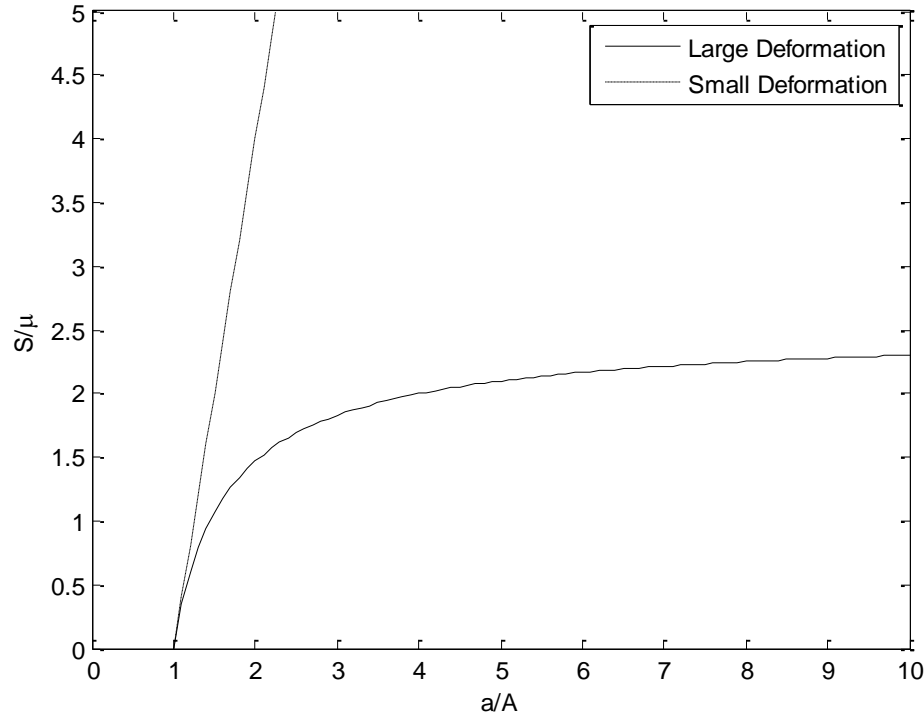
Sketch this result and compare it with the Lamé solution, which assumes Hookean material and infinitesimal deformation. The cavity can expand indefinitely when the remote stress is still finite. The remote stress needed to cause the cavity to expand indefinitely ($a/A \rightarrow \infty$) is called the **cavitation limit**, and is

$$S_c = \frac{5}{2} \mu$$

for a Neo-Hookean material.

- For a review of cavitation, see the iMechanica journal club: Cavitation in Soft Solids, Oscar Lopez-Pamies, May 2010 (<http://imechanica.org/node/8131>).
- S. Kundu and A.J. Crosby, Cavitation and fracture behavior of polyacrylamide hydrogels, *Soft Matter* 5, 3963–3968 (2009).
- Jian Zhu, Tiefeng Li, Shengqiang Cai, and Zhigang Suo. Snap-through expansion of a gas bubble in an elastomer (<http://www.seas.harvard.edu/suo/papers/239.pdf>). The Journal of

Adhesion, in press.



Exercise. Plot the hoop stress at the surface of the cavity as function of a/A . Compare the result with the Lamé solution.

Exercise. Plot S as a function of a/A for a power-law material. Numerical integration might be needed.

Exercise. Study the problem for a cavity in a sphere of material of finite radius.

Exercise. Study the phenomenon of cavitation under the plane strain conditions.

Exercise. In the above formulation, we have used the equilibrium equation formulated in terms of Cauchy stresses as functions of r . Derive the equilibrium equation in terms of the nominal stresses as functions of R . Solve the problem using this alternative formulation.

Helmholtz free energy. A bar, length L and cross-sectional area A in the unstressed state, is stretched to the current state by force P to length l and cross-sectional area a . In the current state, when the bar extends from length l to length $l + \delta l$, the force P does work

$$P\delta l.$$

The bar is in thermal equilibrium with a heat reservoir of a constant temperature. Let F be the Helmholtz free energy of the bar. Recall the definition of the Helmholtz free energy (<http://imechanica.org/node/4878>). When the force P equilibrates with the bar, the work done by the force is converted into the Helmholtz free energy F of the bar, namely,

$$\delta F = P \delta l.$$

Recall the definitions of the nominal stress s and the stretch λ :

$$P = sA, \quad l = \lambda L.$$

Define the free-energy density by

$$W = \frac{\text{energy in the current state}}{\text{volume of the reference state}}.$$

Divide the equation $\delta F = P \delta l$ by the volume of bar in the reference state, AL , and we obtain that

$$\delta W = s \delta \lambda.$$

We can measure experimentally the nominal stress as a function of stretch, $s(\lambda)$, and then integrate the curve $s(\lambda)$ to obtain $W(\lambda)$. Alternatively, we can obtain an expression of $W(\lambda)$ from some theoretical considerations, and then obtain $s(\lambda)$ by

$$s(\lambda) = \frac{dW(\lambda)}{d\lambda}.$$

For the neo-Hookean model, $s(\lambda) = \mu(\lambda - \lambda^{-2})$. Integrating $s(\lambda)$, we obtain that

$$W(\lambda) = \frac{1}{2} \mu (\lambda^2 + 2\lambda^{-1} - 3).$$

The constant of integration is fixed by following the convention that the free energy is set to be zero when the material is unstressed, $W(1) = 0$.

For a nonlinearly elastic material obeying the power law, $\sigma = K\varepsilon^N$, we have $s = K(\log \lambda)^N \lambda^{-1}$, so that

$$W(\lambda) = \frac{K}{N+1} (\log \lambda)^{N+1}.$$

Stability of a state of equilibrium. A bar, of length L and cross-sectional area A in the reference state, is pulled in the current state by a force P . When the force is fixed, the bar may reach a state of equilibrium, of length l . The stretch is defined by $\lambda = l/L$. The material is taken to be elastic, characterized by the free-energy function $W(\lambda)$. What is the value of the stretch in equilibrium?

Is the state of equilibrium stable?

Think of the fixed force P as a weight hanging to the bar. The bar and the weight in combination form a thermodynamic system, with the Helmholtz free energy

$$G = ALW(\lambda) - PL(\lambda - 1).$$

This free energy is a function of the stretch, $G(\lambda)$. We can also call G the Gibbs free energy or the potential energy of the bar.

According to thermodynamics, of all values of the stretch, the equilibrium value minimizes the function $G(\lambda)$. Let $\delta\lambda$ be a small variation. Note the Taylor series:

$$G(\lambda + \delta\lambda) - G(\lambda) = AL \left[\frac{dW(\lambda)}{d\lambda} - \frac{P}{A} \right] \delta\lambda + \frac{1}{2} AL \frac{d^2W(\lambda)}{d\lambda^2} (\delta\lambda)^2.$$

We have retained up to terms quadratic in $\delta\lambda$.

A state of equilibrium is determined by requiring that $dG(\lambda)/d\lambda = 0$, or that

$$\frac{dW(\lambda)}{d\lambda} = \frac{P}{A}.$$

This equation recovers that

$$\frac{dW(\lambda)}{d\lambda} = s.$$

When the force P is prescribed, the nominal stress s is known. When the force equilibrates with the bar, the stretch of the bar, λ , is determined by the above equation, usually a nonlinear algebraic equation.

A state of equilibrium is stable against arbitrary small perturbation in stretch if $d^2G(\lambda)/d\lambda^2 > 0$, or

$$\frac{d^2W(\lambda)}{d\lambda^2} > 0.$$

The critical condition is

$$\frac{d^2W(\lambda)}{d\lambda^2} = 0,$$

or

$$\frac{ds(\lambda)}{d\lambda} = 0.$$

This critical condition is reached at the peak of the curve $s(\lambda)$.

Considère condition reconsidered. For a neo-Hookean material, the free-energy function is

$$W(\lambda) = \frac{1}{2} \mu (\lambda^2 + 2\lambda^{-1} - 3),$$

and the stress-stretch relation is

$$s(\lambda) = \mu (\lambda - \lambda^{-2}).$$

The stress-stretch relation is an increasing function. Consequently, for a given s , the above equation determines a single stretch. The function $W(\lambda)$ is convex, so that every state of equilibrium is stable against small perturbation in stretch.

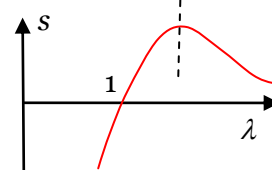
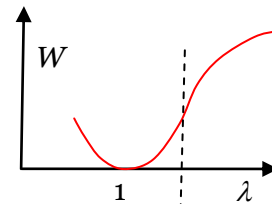
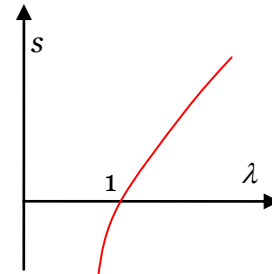
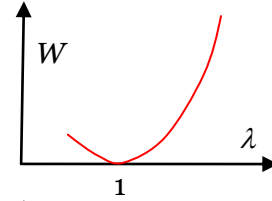
For a power-law material, the free energy function is

$$W(\lambda) = \frac{K}{N+1} (\log \lambda)^{N+1},$$

and the stress-stretch relation is

$$s(\lambda) = K \frac{(\log \lambda)^N}{\lambda}.$$

As the stretch increases from 1, the stress goes up, reaches a peak, and then drops. When the prescribed stress is above the peak, no state of equilibrium exists. When the prescribed stress is below the peak, two states of equilibrium exist. Of the two states of equilibrium, the one with smaller stretch is stable, but the one with larger stretch is unstable. This statement can be verified by checking the sign of $d^2W(\lambda)/d\lambda^2$. Part of the function $W(\lambda)$ is convex, and the other part is concave. The inflection point on the $W(\lambda)$ curve corresponds to the peak on the $s(\lambda)$ curve.



Coexistent phases. Maxwell's rule. Examples include coexistent martensite and austenite of a crystal, and coexistent swollen and collapsed phases of a gel. Such a phenomenon may be modeled by a free-energy function $W(\lambda)$ of a shape shown in the figure. For such a nonconvex $W(\lambda)$, the function $s(\lambda)$ is not monotonic.

In the reference state, Let L be the total length of the bar, and L' be the length of one phase, and L'' be the length of the other phase, so that

$$L' + L'' = L.$$

Let the stretches of the two phases be λ' and λ'' . In the current state, the total length of the bar is l , which is the sum of the lengths of the two phases:

$$\lambda'L' + \lambda''L'' = l.$$

In the current state, the Helmholtz free energy is

$$F = W(\lambda')AL' + W(\lambda'')AL''.$$

According to thermodynamics, when the total length l in the current state is fixed, the two phases equilibrate when the free energy F is minimized.

Denote the average stretch in the bar by

$$\lambda = \frac{l}{L}.$$

When λ is fixed at a value between the tangent points of the common tangent line, the free energy of the bar, F , is minimized if the bar separates into two phases corresponding to the two tangent points. Consequently, when the two phases coexist, the nominal stress in the two phases is the same, denoted by s_t . This thermodynamic statement recovers the familiar fact of mechanical equilibrium. The stress for the phase transition, s_t , is the slope of the common tangent line, so that

$$W'' - W' = s_t(\lambda'' - \lambda').$$

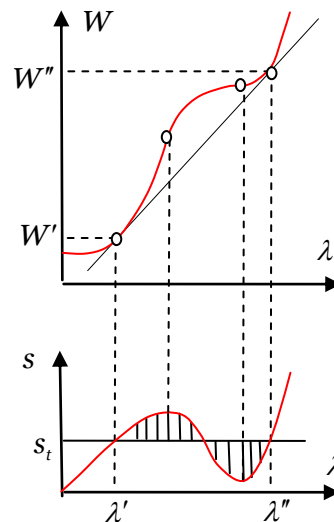
On the (λ, s) plane, the area under the curve $s(\lambda)$ represents the free energy $W(\lambda)$. Inspecting the graph, we can readily see that the above condition means that s_t is at the level such that the two shaded areas are equal. This graphic interpretation is known as *Maxwell's rule*.

Also marked on the $W(\lambda)$ curve are the two inflection points, where

$$\frac{d^2W(\lambda)}{d\lambda^2} = 0.$$

The part of the $W(\lambda)$ curve between the inflection points is concave, where a single phase is unstable. Parts of the $W(\lambda)$ curve are convex, but are between the interval (λ', λ'') . These parts correspond to metastable phases.

Exercise. For the nonlinearly elastic material obeying the power law, $\sigma = K\varepsilon^N$, determine the critical force P_c . Plot the function $G(\lambda)$ for a value of P larger than P_c . Plot the



function $G(\lambda)$ for a value of P smaller than P_c . Comment on these plots.

Exercise. Show that the free energy of the spherical balloon is

$$G = 4\pi R^2 HW - \frac{4\pi}{3} r^3 p.$$

Derive the equation of equilibrium and the critical condition.

Other modes of instability. Of all possible states, the stable state of equilibrium minimizes the free energy. This statement is a basic principle of thermodynamics. But the principle can be hard to implement: it is hard to seek a minimizer among **all** possible states. Say we try to find the tallest human being ever lived. It will not do if we just look among all living Americans. We have to search world-wide. Should we extend our search beyond the Earth? Should we go back to history to look among dead people? Then we do not have good records of people's heights. We will also meet the controversy at what point we wish to call something human.

Thus, in searching for **the state** that minimizes the free energy, we will have to be clear what we mean by **all possible states**. For the bar, we have restricted our search among all states of homogenous deformation. We have excluded states of inhomogeneous deformation. We have formulated a problem solvable in the classroom, but have lost our contact with the experimental observation of necking, which obviously involves inhomogeneous deformation.

If we broaden our search, and include other states as candidates, will we find other states of equilibrium? Yes. Here are some of other models of instability.

Buckle. When we restrict the states to be homogenous, the homogenous state is stable when the force is compressive. When we broaden the search to include bending states, the bar buckles when the compressive force reaches a critical value.

Wrinkle. For bar not very long compared to its width, the homogenous state can be unstable against perturbation of the shape of the surface, even when the compressive force is below the critical load of Euler buckling. M. A. Biot, Surface instability of rubber in compression, *Appl. Sci. Res. A* 12, 168 (1963).

Crease. In all above analysis, the search has been restricted among fields of displacement whose amplitude only slightly deviates from the homogeneous state. Experiments have shown that a block of rubber form creases at even a lower compressive forces. A. N. Gent, I. S. Cho, Surface instabilities in compressed or bent rubber blocks, *Rubber Chemistry and Technology* 72, 253 (1999). This mode of instability has been analyzed recently by E. Hohlfield and L.

Mahadevan (<http://www.seas.harvard.edu/softmat/downloads/2010-05.pdf>).

In creasing, the amplitude of the field deviates greatly from the homogenous state, even though the spatial extent of an initial crease can be small. This mode of instability can also be analyzed by minimizing the free energy (Hong, Zhao, Suo, Applied Physics Letters 95, 111901, 2009, <http://www.seas.harvard.edu/suo/papers/221.pdf>).



Rupture. A rubber band in tension does not form a neck, but does rupture when the stretch is very large.

Vaporization. If we wait long enough, all solids will oxidize, vaporize, or further decompose to more elementary particles. Do we wish to include such states of matter in our search for the energy minimizer?

An alternative formulation of the theory of elasticity. In the reference state, a bar occupies the interval $(0, L)$. Name each material particle by its coordinate X when the bar is in the reference state. In the current state at time t , the material particle X moves to a place with coordinate

$$x = x(X, t).$$

In the current state, the bar is subject to a field of body force $B(X, t)$, is fixed at one end $x(0, t) = 0$, and is subject to a force $P(t)$ at the other end.

Let $\delta x(X)$ be a field of virtual displacement. The field satisfies $\delta x(X) = 0$ and is otherwise arbitrary. Associated with the field of virtual displacement is the field of virtual stretch $\delta \lambda = d(\delta x) / dX$. The virtual work done by the external forces and the inertial force is

$$\int_0^L A(B - \rho \ddot{x}) \delta x dX + P \delta x(L).$$

As a material model, we prescribe a free-energy function $W(\lambda)$. Associated with the field of virtual displacement, define the virtual change in the free energy density by

$$\delta W = \frac{dW(\lambda)}{d\lambda} \delta \lambda.$$

We stipulate that the virtual change of the free energy in the bar equals the

virtual work:

$$\int_0^L A \delta W dX = \int_0^L A (B - \rho \ddot{x}) \delta x dX + P \delta x(L).$$

This equation is the basis for the finite element method.

Note that

$$\begin{aligned} \int_0^L A \delta W dX &= \int_0^L A \frac{dW}{d\lambda} \frac{d\delta x}{dX} dX \\ &= A \frac{dW}{d\lambda} \delta x(L) - \int_0^L \frac{d}{dX} \left(A \frac{dW}{d\lambda} \right) \delta x dX \end{aligned}$$

Comparing the above two equations, and insisting they hold for arbitrary field of virtual displacement, we obtain that

$$\begin{aligned} A \frac{dW}{d\lambda} &= P, \quad \text{at } X = L \\ \frac{d}{dX} \left(A \frac{dW}{d\lambda} \right) + AB &= \rho \ddot{x}, \quad \text{for } X \in (0, L) \end{aligned}$$

These equations recover what we have known once we identify

$$\frac{dW(\lambda)}{d\lambda} = s.$$

Multiaxial stress. We next explore states of multiaxial stress. Of course, the only way to really know stress-strain relations is to run tests, but tests alone would be too time-consuming and quickly become impractical. We'll have to reduce the number of tests by some approximations. The art of making such compromise between accuracy and labor is known as formulating **constitutive models**. As an example, here we begin to describe this art for rubbers.

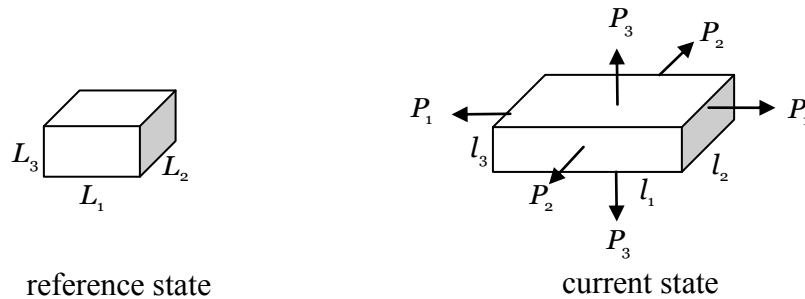
Hyperelastic materials. We can represent a material particle by a rectangular block cut in the orientation of the three principal stresses. Let the block be stretched in the three directions by λ_1 , λ_2 and λ_3 , and the corresponding nominal stresses be s_1 , s_2 and s_3 . The complete stress-strain relations involve three functions $s_1(\lambda_1, \lambda_2, \lambda_3)$, $s_2(\lambda_1, \lambda_2, \lambda_3)$ and $s_3(\lambda_1, \lambda_2, \lambda_3)$. In general, we can run tests to determine these functions. However, as we indicated above, running test alone would be too time-consuming. Instead, we will formulate a constitutive model on the basis that, in equilibrium, the work done by the forces is all stored in the block as energy.

Consider a rectangular block of a material, lengths L_1 , L_2 and L_3 in a

reference state. In a current state, the block is subject to forces P_1 , P_2 and P_3 on its faces, and is deformed into a block of lengths l_1 , l_2 and l_3 . When lengths in the current state change, each of the forces does work, respectively, $P_1\delta l_1$, $P_2\delta l_2$, $P_3\delta l_3$. We assume that the sum of the work done by the external forces equals the change in the free energy. Let F be the Helmholtz free energy of the block in the current state. When the block and the forces are in a state of equilibrium, subject to a small variation of the sides of the block, the work done by the forces equals the change in the free energy:

$$\delta F = P_1\delta l_1 + P_2\delta l_2 + P_3\delta l_3.$$

This condition of equilibrium holds for arbitrary and independent small variations δl_1 , δl_2 , δl_3 .



Let W be nominal density of the free energy, namely, the free energy of the block in the current state divided by the volume of the block in the reference state,

$$W = \frac{F}{L_1 L_2 L_3}.$$

Define the nominal stresses by

$$s_1 = \frac{P_1}{L_2 L_3}, s_2 = \frac{P_2}{L_3 L_1}, s_3 = \frac{P_3}{L_1 L_2}.$$

Define the stretches by

$$\lambda_1 = \frac{l_1}{L_1}, \lambda_2 = \frac{l_2}{L_2}, \lambda_3 = \frac{l_3}{L_3}.$$

Dividing the condition of equilibrium by the volume of the block in the reference state, $L_1 L_2 L_3$, and recalling the definitions of nominal stress and stretch, we obtain that

$$\delta W = s_1 \delta \lambda_1 + s_2 \delta \lambda_2 + s_3 \delta \lambda_3.$$

This condition of equilibrium holds for arbitrary and independent variations $\delta\lambda_1, \delta\lambda_2, \delta\lambda_3$.

As a material model, we assume that the free-energy density is a function of the three stretches:

$$W = W(\lambda_1, \lambda_2, \lambda_3).$$

According calculus,

$$\delta W = \frac{\partial W(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_1} \delta \lambda_1 + \frac{\partial W(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_2} \delta \lambda_2 + \frac{\partial W(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_3} \delta \lambda_3.$$

A comparison of the two expressions for δW , we obtain that

$$\left[s_1 - \frac{\partial W(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_1} \right] \delta \lambda_1 + \left[s_2 - \frac{\partial W(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_2} \right] \delta \lambda_2 + \left[s_3 - \frac{\partial W(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_3} \right] \delta \lambda_3 = 0$$

This condition of equilibrium holds for arbitrary and independent variations $\delta\lambda_1, \delta\lambda_2, \delta\lambda_3$. Consequently, the stresses equal the partial derivatives:

$$s_1 = \frac{\partial W(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_1}, \quad s_2 = \frac{\partial W(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_2}, \quad s_3 = \frac{\partial W(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_3}.$$

An elastic material whose stress-strain relation is derivable from a free-energy function is known as a *hyperelastic material*. By contrast, an elastic material whose stress-strain relation is not derivable from a free-energy function is called a *hypoelastic material*.

The nominal stress on one face is

$$s_1 = \frac{P_1}{L_2 L_3}.$$

The true stress on the same face is

$$\sigma_1 = \frac{P_1}{l_2 l_3}.$$

Consequently, the true stress relates to the nominal stress by

$$\sigma_1 = \frac{s_1}{\lambda_2 \lambda_3}.$$

The true stress is derivable from the strain-energy function:

$$\sigma_1 = \frac{\partial W(\lambda_1, \lambda_2, \lambda_3)}{\lambda_2 \lambda_3 \partial \lambda_1}.$$

The other two components of true stress can be similarly obtained. Note that the true stress is not work-conjugate to the stretch.

Incompressible, isotropic, hyperelastic material. When a

material undergoes large deformation, the amount of volumetric deformation is often small compared to the overall deformation. Consequently, we may neglect the volumetric deformation, and assume that the material is *incompressible*.

A block of a material, of lengths L_1 , L_2 and L_3 in the undeformed state, is deformed into a rectangle of lengths l_1 , l_2 and l_3 . If the material is incompressible, the volume of the block must remain unchanged, namely, $L_1 L_2 L_3 = l_1 l_2 l_3$, or

$$\lambda_1 \lambda_2 \lambda_3 = 1.$$

The incompressibility places a constraint among the three stretches: they cannot vary independently. We may regard λ_1 and λ_2 as independent variables, so that

$$\lambda_3 = (\lambda_1 \lambda_2)^{-1}.$$

Taking differential of the constraint $\lambda_3 = (\lambda_1 \lambda_2)^{-1}$, we obtain that

$$\delta \lambda_3 = \lambda_1^{-2} \lambda_2^{-1} \delta \lambda_1 + \lambda_1^{-1} \lambda_2^{-2} \delta \lambda_2.$$

Inserting the assumption of incompressibility into the condition of equilibrium, $\delta W = s_1 \delta \lambda_1 + s_2 \delta \lambda_2 + s_3 \delta \lambda_3$, we obtain that

$$\delta W = (s_1 - \lambda_1^{-2} \lambda_2^{-1} s_3) \delta \lambda_1 + (s_2 - \lambda_2^{-2} \lambda_1^{-1} s_3) \delta \lambda_2.$$

This condition of equilibrium holds for arbitrary and independent variations $\delta \lambda_1$ and $\delta \lambda_2$.

As a material model, we assume that the free-energy density is a function of the two stretches:

$$W = W(\lambda_1, \lambda_2).$$

According calculus,

$$\delta W = \frac{\partial W(\lambda_1, \lambda_2)}{\partial \lambda_1} \delta \lambda_1 + \frac{\partial W(\lambda_1, \lambda_2)}{\partial \lambda_2} \delta \lambda_2.$$

A comparison of the two expressions for δW , we obtain that

$$\left[s_1 - \lambda_1^{-2} \lambda_2^{-1} s_3 - \frac{\partial W(\lambda_1, \lambda_2)}{\partial \lambda_1} \right] \delta \lambda_1 + \left[s_2 - \lambda_2^{-2} \lambda_1^{-1} s_3 - \frac{\partial W(\lambda_1, \lambda_2)}{\partial \lambda_2} \right] \delta \lambda_2 = 0$$

This condition of equilibrium holds for arbitrary and independent variations $\delta \lambda_1, \delta \lambda_2$. Consequently, once the function $W(\lambda_1, \lambda_2)$ is determined, the stress-stretch relation is given by differentiation:

$$s_1 - \lambda_1^{-2} \lambda_2^{-1} s_3 = \frac{\partial W(\lambda_1, \lambda_2)}{\partial \lambda_1},$$

$$s_2 - \lambda_2^{-2} \lambda_1^{-1} s_3 = \frac{\partial W(\lambda_1, \lambda_2)}{\partial \lambda_2}.$$

These relations, together with the incompressibility condition $\lambda_1 \lambda_2 \lambda_3 = 1$, replace Hooke's law and serve as the stress-strain relations for incompressible, isotropic and hyperelastic materials.

For incompressible materials, the true stresses relate to the nominal stresses as

$$\sigma_1 = \lambda_1 s_1, \quad \sigma_2 = \lambda_2 s_2, \quad \sigma_3 = \lambda_3 s_3.$$

Consequently, the stress-strain relations become

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial W(\lambda_1, \lambda_2)}{\partial \lambda_1},$$

$$\sigma_2 - \sigma_3 = \lambda_2 \frac{\partial W(\lambda_1, \lambda_2)}{\partial \lambda_2}.$$

Because the material is incompressible, once the stretches are known, the material model leaves the hydrostatic stress undetermined.

The function $W(\lambda_1, \lambda_2)$ can be determined by subjecting a sheet of a rubber under various states of biaxial stress. The form of the function is sometimes inspired by theoretical considerations. Here are some often used forms.

Neo-Hookean model. This model is specified by the energy-density function:

$$W = \frac{\mu}{2} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3).$$

The material is also taken to be incompressible, $\lambda_1 \lambda_2 \lambda_3 = 1$.

Recall that

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial W(\lambda_1, \lambda_2)}{\partial \lambda_1}.$$

Inserting the constraint $\lambda_3 = (\lambda_1 \lambda_2)^{-1}$, we obtain the stress-stretch relation:

$$\sigma_1 - \sigma_3 = \mu (\lambda_1^2 - \lambda_3^2).$$

Similarly, we obtain that

$$\sigma_2 - \sigma_3 = \mu (\lambda_2^2 - \lambda_3^2).$$

A neo-Hookean material is characterized by a single elastic constant, μ . The constant may be determined experimentally. For example, consider a bar in a state of uniaxial stress:

$$\sigma_1 = \sigma, \quad \sigma_2 = \sigma_3 = 0.$$

Let the stretch along the axis of loading be $\lambda_1 = \lambda$. Incompressibility dictates that the stretches in the directions transverse to the loading axis be $\lambda_2 = \lambda_3 = \lambda^{-1/2}$. Inserting into the above stress-stretch relation, we obtain the relation under the uniaxial stress:

$$\sigma = \mu(\lambda^2 - \lambda^{-1}).$$

Recall that stretch relates to the engineering strain as

$$\lambda = l/L = 1 + e.$$

When the strain is small, namely, $|e| \ll 1$, the above stress-stretch relation reduces to

$$\sigma = 3\mu e.$$

Thus, we interpret 3μ as Young's modulus and μ the shear modulus.

The form of strain-energy function of Neo-Hookean materials has also emerged from a model in statistical mechanics. The model gives $\mu = NkT$, where N is the number of polymer chains per unit volume, and kT the temperature in units of energy.

- L.R.G. Treloar, *The Physics of Rubber Elasticity*, Third Edition, Oxford University Press, 1975.

Gent model. In an elastomer, each individual polymer chain has a finite contour length. When the elastomer is subject no loads, the polymer chains are coiled, allowing a large number of conformations. Subject to loads, the polymer chains become less coiled. As the loads increase, the end-to-end distance of each polymer chain approaches the finite contour length, and the elastomer approaches a limiting stretch. On approaching the limiting stretch, the elastomer stiffens steeply. This effect is absent in the neo-Hookean model, but is represented by the *Gent model*:

$$W = -\frac{\mu J_{\text{lim}}}{2} \log \left(1 - \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3}{J_{\text{lim}}} \right).$$

where μ is the small-stress shear modulus, and J_{lim} is a constant related to the limiting stretch. The stretches are restricted as $0 \leq (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)/J_{\text{lim}} < 1$. When $(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)/J_{\text{lim}} \rightarrow 0$, the Taylor expansion of the Gent model recovers the neo-Hookean mode. When $(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)/J_{\text{lim}} \rightarrow 1$, the free energy diverges, and the elastomer approaches the limiting stretch.

The material is also taken to be incompressible. The stress-stretch

relations are

$$\sigma_1 - \sigma_3 = \frac{\mu(\lambda_1^2 - \lambda_3^2)}{1 - \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3}{J_{\text{lim}}}},$$

$$\sigma_2 - \sigma_3 = \frac{\mu(\lambda_2^2 - \lambda_3^2)}{1 - \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3}{J_{\text{lim}}}}.$$

- Gent, A. N., A new constitutive relation for rubber. Rubber Chemistry and Technology, 1996, 69: 59-61.

Model a polymer as a freely-jointed chain. First consider an individual polymer. Let Λ be the stretch of the polymer, namely, the end-to-end length of the stretched polymer r divided by that of the released polymer r_0 , namely,

$$\Lambda = \frac{r}{r_0}$$

The Helmholtz free energy of the polymer, w , is a function of the stretch, namely,

$$w = w(\Lambda).$$

Consider the model of freely-jointed chain of Kuhn and Grun. In this model, a polymer chain is modeled by a sequence of links capable of free rotation relative to each other. At a finite temperature, the relative rotations of the links allow the polymer chain to fluctuate among a large number of conformations. The statistics of the conformations gives the force-stretch behavior of the polymer chain

$$\Lambda = \sqrt{n} \left(\frac{1}{\tanh \zeta} - \frac{1}{\zeta} \right)$$

where n is the number of links per polymer chain, and ζ the normalized force in the polymer chain.

The Helmholtz free energy of the polymer is

$$w = nkT \left(\frac{\zeta}{\tanh \zeta} - 1 + \log \frac{\zeta}{\sinh \zeta} \right)$$

where kT is the temperature in the unit of energy.

- W. Kuhn and F. Grun, Kolloidzshr. **101**, 248 (1942).
- L.R.G. Treloar, The Physics of Rubber Elasticity, Third Edition, Oxford University Press, 1975.

Arruda-Boyce eight-chain model. Next we consider a network of polymers, crosslinked by covalent bonds. Many models exist in the literature to derive the behavior of the network from that of the individual polymers. In the Arruda and Boyce model, the network is represented by many cells, each cell by a rectangle, and the polymers in the cell by the eight half diagonals. When the network is unstressed, the cell is a unit cube, and the length of a half diagonal is $\sqrt{3}/2$. When the network is stretched, the cell is a rectangle with sides λ_1 , λ_2 and λ_3 , and the length of a half diagonal is $(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{1/2}/2$. Consequently, the end-to-end length of each chain is

$$\Lambda = \frac{1}{\sqrt{3}} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{1/2}.$$

Let v be the volume per link, and nv be the volume per chain. The free energy per unit volume of the network is

$$W = \frac{w(\Lambda)}{nv}.$$

The eight-chain model is described by a free energy function of the form $W(\Lambda)$.

Next we represent each chain by using the freely-jointed chain model. The Helmholtz free energy per unit volume of the elastomer is

$$W = \frac{kT}{v} \left(\frac{\zeta}{\tanh \zeta} - 1 + \log \frac{\zeta}{\sinh \zeta} \right)$$

These Equations defines the function $W(\lambda_1, \lambda_2)$ using two intermediate parameters: the stretch Λ in each chain and the normalized force ζ in each chain. In one limit, $\zeta \rightarrow \infty$, the chain approaches the limiting stretch, $\Lambda \rightarrow \sqrt{n}$. In the other limit, $\zeta \rightarrow 0$, the chain coils much below the limiting stretch, $\Lambda \ll \sqrt{n}$, and the model reduces to $W = (kT/6v)\zeta^2$ and $\Lambda = (\sqrt{n}/3)\zeta$, which recovers the neo-Hookean model, with the small-strain shear modulus $\mu = kT/(vn)$.

The material is also taken to be incompressible, $\lambda_1 \lambda_2 \lambda_3 = 1$. Recall that

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial W(\lambda_1, \lambda_2)}{\partial \lambda_1}.$$

According to calculus, we write

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{dW(\zeta)}{d\zeta} \left(\frac{d\Lambda(\zeta)}{d\zeta} \right)^{-1} \frac{\partial \Lambda(\lambda_1, \lambda_2)}{\partial \lambda_1}.$$

A direct calculation gives that

$$\sigma_1 - \sigma_3 = \frac{kT\zeta(\lambda_1^2 - \lambda_3^2)}{3\nu\sqrt{n}\Lambda}.$$

We can similarly obtain that

$$\sigma_2 - \sigma_3 = \frac{kT\zeta(\lambda_2^2 - \lambda_3^2)}{3\nu\sqrt{n}\Lambda}.$$

Like the Gent model, the model of freely-joined chain can also capture the extension limit.

- E.M. Arruda and M.C. Boyce, A three dimensional constitutive model for the large stretch behavior of rubber elastic materials. *Journal of the Mechanics and Physics of Solids* 41, 389-412 (1993).

Mooney model. In this model, the free energy is fit to the expression

$$W = c_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + c_2(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3).$$

- M. Mooney (1940) A theory of large elastic deformation, *Journal of Applied Physics* 11, 582-592.

Ogden model. In this model, the free energy density is fit to a series of more terms

$$W = \sum_n \frac{\mu_n}{\alpha_n} (\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3),$$

where α_n may have any values, positive or negative, and are not necessarily integers, and μ_n are constants. The stress-stretch relations are

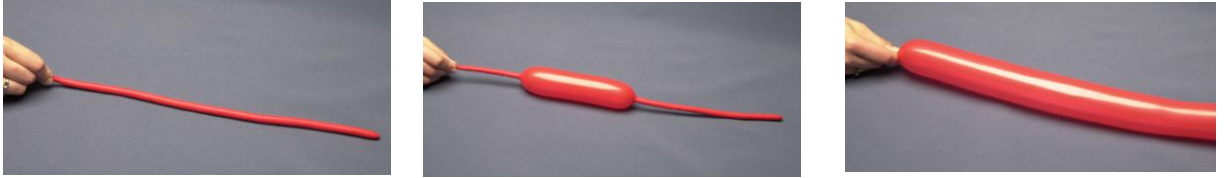
$$\begin{aligned}\sigma_1 - \sigma_3 &= \sum_n \mu_n (\lambda_1^{\alpha_n} - \lambda_3^{\alpha_n}), \\ \sigma_2 - \sigma_3 &= \sum_n \mu_n (\lambda_2^{\alpha_n} - \lambda_3^{\alpha_n}).\end{aligned}$$

- R.W. Ogden (1972) Large deformation isotropic elasticity – on the correlation of theory and experiment for incompressible rubberlike solids. *Proceedings of the Royal Society of London A* 326, 567-583.

Reading. Erwan Verron (<http://imechanica.org/node/4167>), a bibliography of the literature on the mechanics of rubbers up to 2003.

Exercise. Study the inflation of a spherical balloon using various material models. See a review of calculations and experiments on p. 239 of G.A. Holzapfel, *Nonlinear Solid Mechanics*, Wiley, 2000).

Exercise. Study the inflation of a cylindrical balloon. Can your solution interpret the observation of a party balloon?



<http://www.doitpoms.ac.uk/tlplib/bioelasticity/demo.php>

Reading. Chater, E., Hutchinson, J.W., On the Propagation of Bulges and Buckles. *Journal of Applied Mechanics*, 51, 1-9 (1984).

<http://www.seas.harvard.edu/hutchinson/papers/377.pdf>

Stability of a finite homogenous deformation against perturbation of infinitesimal homogenous deformation. Consider a cube of an incompressible material subject to a hydrostatic stress. An obvious state of equilibrium is that the cube remains undeformed. If the hydrostatic stress is fixed as a true stress, the undeformed state is stable against homogenous perturbation of any magnitude. This general conclusion is understood as follows. Because the material is incompressible, the hydrostatic stress does no work as the cube deforms. Consequently, thermodynamics requires that a stable state of equilibrium minimize the Helmholtz free energy. In reaching the above conclusion, we have assumed that the Helmholtz free energy of the material does increase when the material deforms.

The situation is more interesting if, instead of fixed true stresses, fixed forces of equal magnitude are applied to the cube in the three directions. The undeformed state is still a state of equilibrium. The undeformed state, however, can be unstable. In this case, the forces do net work when the cube deforms. Thermodynamics requires that a stable state of equilibrium minimizes the Gibbs free energy:

$$G = L^3 W(\lambda_1, \lambda_2) - PL \left(\lambda_1 + \lambda_2 + \frac{1}{\lambda_1 \lambda_2} - 3 \right).$$

The Gibbs free energy is a function of the two stretches, $G(\lambda_1, \lambda_2)$. We assume that $W(1,1) = 0$, and $W(\lambda_1, \lambda_2) > 0$ when the cube deforms.

When the cube is undeformed, $\lambda_1 = \lambda_2 = 1$, the Gibbs function vanishes. When the cube deforms, the term W increases the Gibbs function, but the term due to the applied forces always reduces the Gibbs function, provided the applied forces are tensile, $P > 0$. Consequently, when the applied force P is tensile and is

of a large enough magnitude, the undeformed state will no longer minimize the Gibbs function, and the cube will deform.

Pursuing this matter further, we examine the Taylor expansion:

$$G(\lambda_1 + \delta\lambda_1, \lambda_2 + \delta\lambda_2) - G(\lambda_1, \lambda_2) = \frac{\partial G(\lambda_1, \lambda_2)}{\partial \lambda_1} \delta\lambda_1 + \frac{\partial G(\lambda_1, \lambda_2)}{\partial \lambda_2} \delta\lambda_2 \\ + \frac{\partial^2 G(\lambda_1, \lambda_2)}{2\partial \lambda_1^2} (\delta\lambda_1)^2 + \frac{\partial^2 G(\lambda_1, \lambda_2)}{2\partial \lambda_2^2} (\delta\lambda_2)^2 + \frac{\partial^2 G(\lambda_1, \lambda_2)}{\partial \lambda_1 \partial \lambda_2} (\delta\lambda_1 \delta\lambda_2)$$

Setting the first derivatives to zero, we obtain that

$$\frac{\partial G(\lambda_1, \lambda_2)}{\partial \lambda_1} = \mathbf{0}, \quad \frac{\partial G(\lambda_1, \lambda_2)}{\partial \lambda_2} = \mathbf{0}.$$

This is a pair of nonlinear algebraic equations that determine the two stretches of any state of equilibrium. To determine the stretches in equilibrium, you will need to specify a material model, $W(\lambda_1, \lambda_2)$, and a level of the nominal stress $s = P/L^2$. Multiple deformed states of equilibrium may exist.

For a state of equilibrium to be stable against the infinitesimal perturbation, the quadratic form

$$\frac{\partial^2 G(\lambda_1, \lambda_2)}{2\partial \lambda_1^2} (\delta\lambda_1)^2 + \frac{\partial^2 G(\lambda_1, \lambda_2)}{2\partial \lambda_2^2} (\delta\lambda_2)^2 + \frac{\partial^2 G(\lambda_1, \lambda_2)}{\partial \lambda_1 \partial \lambda_2} (\delta\lambda_1 \delta\lambda_2)$$

must be positive-definite for any infinitesimal change in the stretches, $\delta\lambda_1$ and $\delta\lambda_2$. This condition, according to a theorem in linear algebra, requires that the following three conditions be satisfied:

$$\frac{\partial^2 G(\lambda_1, \lambda_2)}{2\partial \lambda_1^2} > \mathbf{0} \\ \frac{\partial^2 G(\lambda_1, \lambda_2)}{2\partial \lambda_2^2} > \mathbf{0} \\ \left[\frac{\partial^2 G(\lambda_1, \lambda_2)}{\partial \lambda_1^2} \right] \left[\frac{\partial^2 G(\lambda_1, \lambda_2)}{\partial \lambda_2^2} \right] > \left[\frac{\partial^2 G(\lambda_1, \lambda_2)}{\partial \lambda_1 \partial \lambda_2} \right]^2$$

These conditions need be verified for each of the state of equilibrium to confirm its stability.

Exercise. For the neo-Hookean material, calculate the critical force on the cube for the undeformed state to be unstable. Determine the states of equilibrium when the applied force exceeds the critical level. (R.S. Rivlin, Large elastic deformation of isotropic materials. II. Some uniqueness theorems for pure homogeneous deformation. Philos. Trans. Roy. Soc. Lond. A 240, 491-508, 1948)

Exercise. Modify the above model for a sheet of a rubber subject to equal biaxial forces. Show that the stretches in two directions in the plane of sheet can be unequal. (E.A. Kearsley, Asymmetric stretching of a symmetrically loaded elastic sheet. *Int. J. Solids Structures* 22, 111-119, 1986)