Finite Deformation: General Theory

The notes on finite deformation have been divided into two parts: special cases (http://imechanica.org/node/5065) and general theory. In class I start with special cases, and then sketch the general theory. But the two parts can be read in any order.

Subject to loads, a body deforms. We would like to develop a theory to evolve this deformation in time. In continuum mechanics, we model the body by a field of particles, and update the positions of the particles by using an equation of motion. We formulate the equation of motion by mixing the following ingredients:

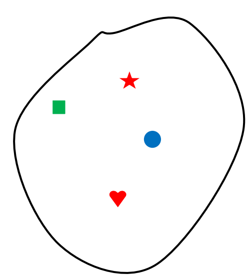
- kinematics of deformation,
- conservation of mass,
- conservation of linear momentum,
- conservation of angular momentum,
- models of materials.

Each of these ingredients has alternative, but equivalent, mathematical representations. To focus on essential ideas, we will first adopt one set of representations, using nominal quantities. At the end of the notes, we will sketch some of the alternatives.

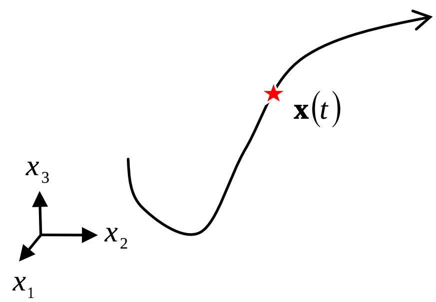
Model a body by a field of material particles. A body is made of atoms, each atom is made of electrons, protons and neutrons, and each proton or neutron is made of... This kind of description is too detailed. We will not go very far in helping the engineer if we keep thinking of a bridge as a pile of atoms.

Instead, we will develop a continuum theory. This theory models the body by a field of *material particles*, or *particles* for brevity. Each particle consists of many atoms. The clouds of electrons deform, and the protons jiggle, all at a maddeningly high frequency. The material particle, however, represents the collective behavior of many atoms, over a size much larger than the radius of individual atoms, and over a time much longer than the jiggle of individual atoms.

A common method to devise a material model is to regard each material particle as a small specimen, undergoing homogenous deformation. The model takes input from the experiments with such specimens.



Subject to loads, the body deforms in a three-dimensional space. The space consists of *places*, labeled by coordinates $\mathbf{x}$. At a given time t, each material particle in the body occupies a place in the space. As time progresses, the material particle moves from one place to another. The trajectory of the material particle is described by the place of the particle as a function of time, $\mathbf{x}(t)$.



When a body deforms, does each material particle preserve its identity? We have tacitly assumed that, when a body deforms, each material particle preserves its identity. Whether this assumption is valid can be determined by experiments. For example, we can paint a grid on the body. After deformation, if the grid is distorted but remains intact, then we say that the deformation preserves the identity of each particle. If, however, after the deformation the grid disintegrates, we should not assume that the deformation preserves the identity of each particle.

Whether a deformed body preserves the identity of the particle is subjective, depending on the size and time scale over which we look at the body. A rubber, for example, consists of cross-linked long molecules. If our grid is over a size much larger than the individual molecular chains, then deformation will not cause the grid to disintegrate. By contrast, a liquid consists of molecules that can change neighbors. A grid painted on a body of liquid, no matter how coarse the grid is, will disintegrate over a long enough time. Similar remarks may be made for metals undergoing plastic deformation. Also, in many situations, the body will grow over time. Examples include growth of cells in a tissue, and growth of thin films when atoms diffuse into the films. The combined growth and deformation clearly does not preserve the identity of each material particle.

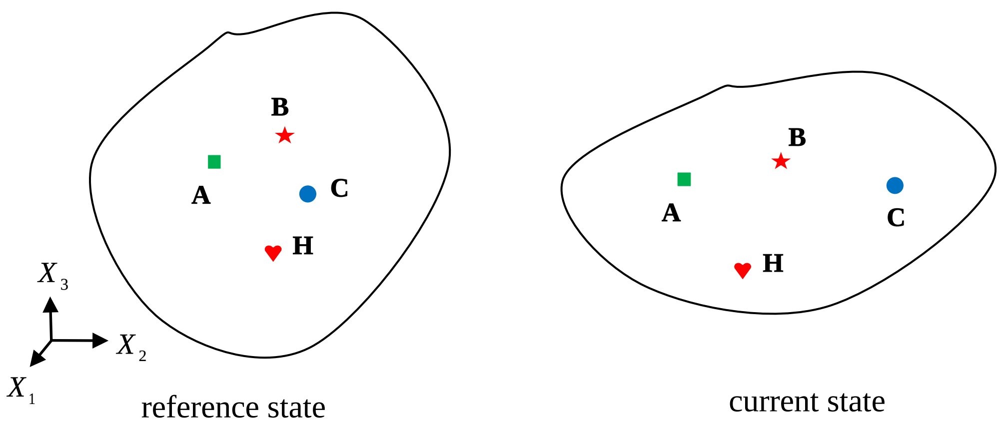
In these notes, we will assume that the identity of each material particle is preserved as the body deforms.

Name a material particle by the coordinate of the place occupied by the material particle when the body is in a reference state. We can name a material particle any way we like. For example, we often name a material particle by using an English letter, a Chinese character, or a colored symbol—a red star for example. When dealing with a large number of material particles, we need a systematic naming scheme. For example, we name each material particle by the coordinate $\mathbf{X}$ of the place occupied by the material particle when the body is in a particular state.

We call this particular state of the body the *reference state*. We will use the phrase "the particle $\mathbf{X}$" as shorthand for "the material particle that occupies the place with coordinate $\mathbf{X}$ when the body is in the reference state". In addition to being systematic, naming material particles by coordinates has another merit. Once we know the name of one material particle, we know the names of all its neighbors, and we can apply calculus.

Often we choose the reference state to be the state when the body is unstressed. However, even without external loading, a body may be under a field of residual stress. Thus, we may not be able to always set the reference state as the unstressed state. Rather, any state of the body may be used as a reference state.

Indeed, the reference state need not be an actual state of the body, and can be a hypothetical state of the body. For example, we can use a flat plate as a reference state for a shell, even if the shell is always curved and is never flat. To enable us to use differential calculus, all that matters is that material particles can be mapped from the reference state to any actual state by a 1-to-1 smooth function.

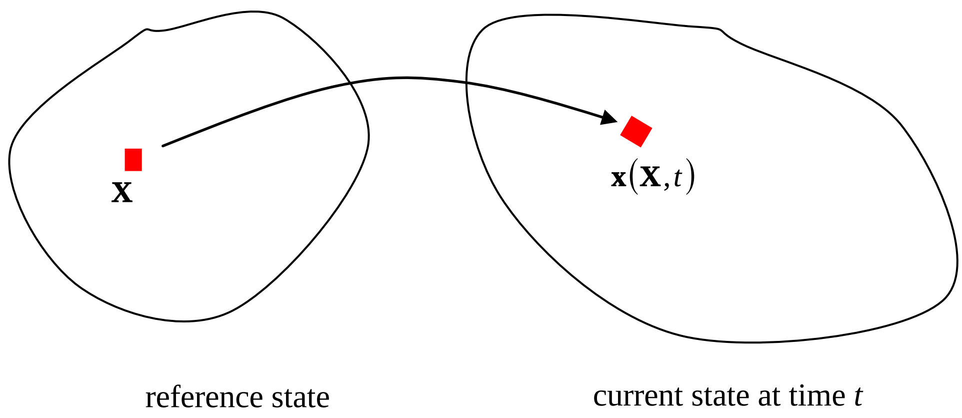


Field of deformation. Now we are given a body in a three-dimensional space. We have set up a system of coordinates in the space, and have chosen a reference state of the body to name material particles. When the body is in the reference state, a material particle occupies a place whose coordinate is $\mathbf{X}$. At time t, the body deforms to a *current state*, and the material particle $\mathbf{X}$ moves to a place whose coordinate is $\mathbf{x}$. The time-dependent field

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$$

describes the history of deformation of the body. The domain of this function is the coordinates of material particles when the body is in the reference state, as well as the time. The range of this function is the coordinates of the places occupied by the material particles. A central aim of continuum mechanics is to evolve the field of deformation $\mathbf{x}(\mathbf{X},t)$ by developing an equation of motion.

The function $\mathbf{x}(\mathbf{X},t)$ has two independent variables: $\mathbf{X}$ and t. The two variables can change independently. We next examine their changes separately.



Exercise. Give a pictorial interpretation of the following field of deformation:

$$x_1 = X_1 + X_2 \tan\gamma(t)$$
$$x_2 = X_2$$
$$x_3 = X_3$$

Compare the above field with another field of deformation:

$$x_1 = X_1 + X_2 \sin\gamma(t)$$
$$x_2 = X_2 \cos\gamma(t)$$
$$x_3 = X_3$$

Displacement, velocity, and acceleration of a material particle. At time t, the material particle $\mathbf{X}$ occupies the place $\mathbf{x}(\mathbf{X},t)$. At a slightly later time $t+\delta t$, the same material particle $\mathbf{X}$ occupies a different place $\mathbf{x}(\mathbf{X},t+\delta t)$. During the short time between t and $t+\delta t$, the material particle $\mathbf{X}$ moves by a small displacement:

$$\delta\mathbf{x} = \mathbf{x}(\mathbf{X},t+\delta t) - \mathbf{x}(\mathbf{X},t).$$

The velocity of the material particle $\mathbf{X}$ at time t is defined as

$$\mathbf{v} = \frac{\mathbf{x}(\mathbf{X},t+\delta t) - \mathbf{x}(\mathbf{X},t)}{\delta t},$$

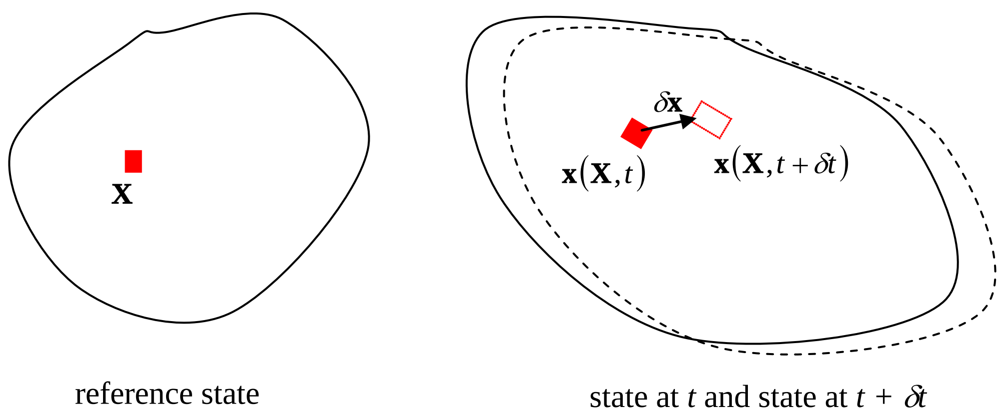
or,

$$\mathbf{v} = \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}.$$

The velocity is a time-dependent field, $\mathbf{v}(\mathbf{X},t)$. The acceleration of the material particle $\mathbf{X}$ at time *t* is

$$\mathbf{a}(\mathbf{X},t) = \frac{\partial^2 \mathbf{x}(\mathbf{X},t)}{\partial t^2}.$$

The fields of velocity and acceleration are linear in the field of deformation $\mathbf{x}(\mathbf{X},t)$.



Exercise. Given a field of deformation,

$$x_1 = X_1 + X_2 \sin\gamma(t)$$
$$x_2 = X_2 \cos\gamma(t)$$
$$x_3 = X_3$$

Calculate the fields of velocity and acceleration.

Deformation gradient. We have just interpreted the partial derivative of the function $\mathbf{x}(\mathbf{X},t)$ with respect to *t*. We next interpret the partial derivative of the function $\mathbf{x}(\mathbf{X},t)$ with respect to $\mathbf{X}$. For a bar undergoing homogenous deformation, we have defined the stretch by

$$\text{stretch} = \frac{\text{length in current state}}{\text{length in reference state}}.$$

We now extend this definition to a body undergoing inhomogeneous deformation in three dimensions.

Consider two nearby material particles in the body. When the body is in the reference state, the first particle occupies the place with the coordinate $\mathbf{X}$, and the second particle occupies the place with the coordinate $\mathbf{X} + d\mathbf{X}$. The vector $d\mathbf{X}$ connects the places occupied by the two material particles when the body is in the reference state.

When the body is in the current state at time t, the first material particle occupies the place with the coordinate $\mathbf{x}(\mathbf{X},t)$, and the second material particle occupies the place with the coordinate $\mathbf{x}(\mathbf{X}+d\mathbf{X},t)$.

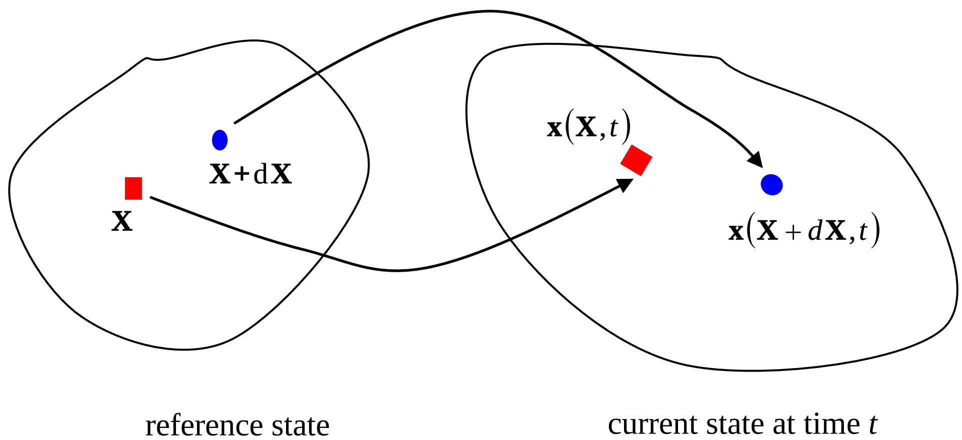
The stretch is generalized to

$$\frac{x_i(\mathbf{X}+d\mathbf{X},t)-x_i(\mathbf{X},t)}{dX_K}.$$

This object represents 9 ratios. We give the object a symbol

$$F_{iK}=\frac{\partial x_i(\mathbf{X},t)}{\partial X_K},$$

and call the object the *deformation gradient*. That is, we have generalized the stretch to a time-dependent field of tensor, $\mathbf{F}(\mathbf{X},t)$. The deformation gradient $\mathbf{F}(\mathbf{X},t)$ is linear in the field of deformation $\mathbf{x}(\mathbf{X},t)$.



Exercise. Given a field of deformation,

$$x_1=X_1+X_2\sin\gamma(t)$$
$$x_2=X_2\cos\gamma(t)$$
$$x_3=X_3$$

Calculate the deformation gradient.

As a body deforms, each material element of line rotates and stretches. Now focus on the vector connecting the two nearby material particles, $\mathbf{X}$ and $\mathbf{X}+d\mathbf{X}$. We call the vector a *material element of line*. When the body is in the reference state, the material element of line is represented by the vector $d\mathbf{X}$. When the body is in the current state at time t, this material element of line is represented by the vector

$$d\mathbf{x} = \mathbf{x}(\mathbf{X} + d\mathbf{X}, t) - \mathbf{x}(\mathbf{X}, t).$$

That is, the same material element of line is represented by the vector $d\mathbf{X}$ when the body is in the reference state, and by the vector $d\mathbf{x}$ when the body is in the current state. These two representations are connected by

$$dx_i = \frac{\partial x_i(\mathbf{X}, t)}{\partial X_K} dX_K .$$

We have adopted the convention of summing over repeated indices.

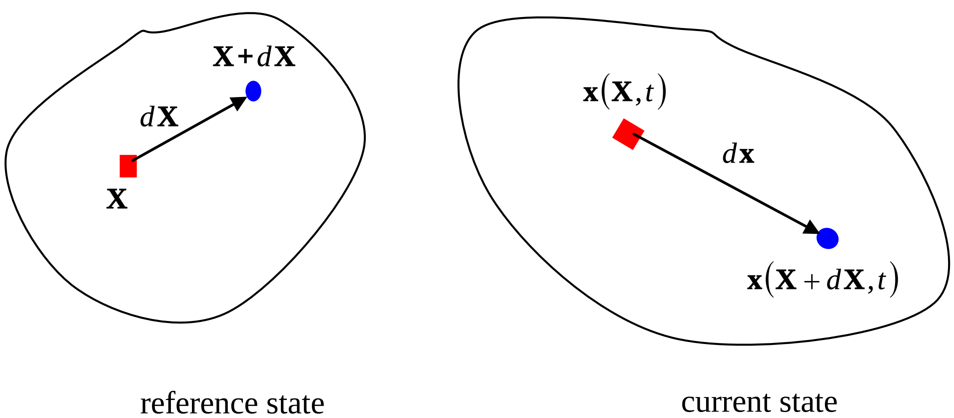
We have called the partial derivative the deformation gradient, so that

$$dx_i = F_{iK}(\mathbf{X}, t) dX_K ,$$

or

$$d\mathbf{x} = \mathbf{F}(\mathbf{X}, t) d\mathbf{X}.$$

Thus, the deformation gradient $\mathbf{F}(\mathbf{X}, t)$ is a linear operator that maps the vector between two nearby material particles in the reference state, $d\mathbf{X}$, to the vector between the same two material particles in the current state, $d\mathbf{x}$.



When the body deforms, the material element of line both stretches and rotates. In the reference state, let dL be the length of the element and $\mathbf{M}$ be the unit vector in the direction of the element, namely,

$$d\mathbf{X} = \mathbf{M} dL .$$

In the current state, let dl be the length of the element and $\mathbf{m}$ be the unit vector in the direction of the element, namely,

$$d\mathbf{x} = \mathbf{m} dl .$$

Thus, the deformation of the body stretches the length of the material element of line from dL to dl, and rotates the direction of the material element from $\mathbf{M}$ to $\mathbf{m}$. Recall that the stretch of the element is defined by

$$\lambda = \frac{dl}{dL}.$$

Combining the above four equations, we obtain that

$$\lambda \mathbf{m} = \mathbf{F}(\mathbf{X},t)\mathbf{M}.$$

In the current state at time *t*, once the field of deformation, $\mathbf{x}(\mathbf{X},t)$, is known, the deformation gradient $\mathbf{F}(\mathbf{X},t)$ can be calculated. Once the direction $\mathbf{M}$ of a material element of line is given when the body is in the reference state, the above expression gives in the current state the direction of the element, $\mathbf{m}(\mathbf{X},t,\mathbf{M})$, and the stretch of the element, $\lambda(\mathbf{X},t,\mathbf{M})$.

Green deformation tensor. We can also obtain an explicit expression for the stretch $\lambda(\mathbf{X},t,\mathbf{M})$. Taking the inner product of the vector $\lambda \mathbf{m} = \mathbf{F}(\mathbf{X},t)\mathbf{M}$, we obtain that

$$\lambda^2 = \mathbf{M}^T\mathbf{C}\mathbf{M}.$$

where

$$\mathbf{C} = \mathbf{F}^T\mathbf{F},$$

or

$$C_{KL} = F_{iK}F_{iL}.$$

The time-dependent field, $\mathbf{C}(\mathbf{X},t)$, known as the *Green deformation tensor*, is symmetric and positive-definite. The formula is colored green.

In the current state at time *t*, once the field of deformation, $\mathbf{x}(\mathbf{X},t)$, is known, the Green deformation tensor $\mathbf{C}(\mathbf{X},t)$ can be calculated. Once $\mathbf{C}(\mathbf{X},t)$ is known, for any material element of line of given direction $\mathbf{M}$, we can calculate the stretch of the element, $\lambda(\mathbf{X},t,\mathbf{M})$. Thus, the Green deformation tensor fully specifies the state of deformation of the material particle.

Exercise. Given a field of deformation:

$$x_1 = X_1 + X_2 \sin\gamma(t)$$
$$x_2 = X_2 \cos\gamma(t)$$
$$x_3 = X_3$$

When the body is in the reference state, a material element of line is in the direction $\mathbf{M} = [0,1,0]^T$. Calculate the stretch and the direction of the element when the body is in the current state at time *t*.

Exercise. Any symmetric and positive-definite matrix has three orthogonal eigenvectors, along with three real and positive eigenvalues. Interpret the geometric significance of the eigenvectors and the eigenvalues of $\mathbf{C}(\mathbf{X},t)$.

Exercise. For the above field of deformation, determine the eigenvectors and eigenvalues of the tensor $\mathbf{C}(\mathbf{X},t)$. Interpret your result.

Polar decomposition. By definition, the deformation gradient is a linear operator that maps one vector to another vector. For any nonsingular linear operator here is a theorem in linear algebra. Let $\mathbf{F}$ be a linear operator. The operator is nonsingular, i.e., $\det\mathbf{F} \neq 0$. The linear operator can be written as a product:

$$\mathbf{F} = \mathbf{RU},$$

where $\mathbf{R}$ is an orthogonal operator, satisfying $\mathbf{R}^T\mathbf{R} = \mathbf{I}$, and $\mathbf{U}$ is a symmetric operator calculated from

$$\mathbf{F}^T\mathbf{F} = \mathbf{U}^2.$$

Writing a linear operator in this way is known as polar decomposition.

Exercise. Prove the theorem of polar decomposition by showing that, for a given nonsingular linear operator $\mathbf{F}$, one can find a unique symmetric operator $\mathbf{U}$ and a unique orthogonal operator $\mathbf{R}$ such that $\mathbf{F} = \mathbf{RU}$.

Exercise. Note that

$$\mathbf{C} = \mathbf{U}^2.$$

Interpret eigenvalues and eigenvectors of $\mathbf{C}$ and $\mathbf{U}$. Give pictorial interpretation of the polar decomposition.

Deformation of a material element of volume. Let $dV = dX_1 dX_2 dX_3$ be an element of volume in the reference state. After deformation, the line element dX_1 becomes $d\mathbf{x}^1$, with components given by

$$dx_i^1 = F_{i1} dX_1.$$

We can write similar relations for $d\mathbf{x}^2$ and $d\mathbf{x}^3$. In the current state, the volume of the element is

$$dv = \left(d\mathbf{x}^1 \times d\mathbf{x}^2\right) \cdot d\mathbf{x}^3.$$

We can confirm that this expression is the same as

$$dv = \det(\mathbf{F})dV \; .$$

Exercise. Given a field of deformation:

$$x_1 = 3 + X_1 + 2X_2 + 3X_3$$
$$x_2 = 1 + X_2$$
$$x_3 = X_1 + 6X_3$$

Observe that this field represents a body undergoing a state of homogeneous deformation. Calculate the ratio dv/dV.

Deformation of a material element of area. Consider a material element of area. When the body is in the reference state, $\mathbf{N}$ is the unit vector normal to the element, dA is the area of the element, and we write the element of area as a vector:

$$d\mathbf{A} = \mathbf{N}dA \; .$$

When the body is in the current state, the same material element of area deforms and reorient, so that $\mathbf{n}$ is the unit vector normal to the element and da is the area of the element, and we write the element of area as a vector:

$$d\mathbf{a} = \mathbf{n}da \; .$$

An element of line $d\mathbf{X}$ in the reference state becomes $d\mathbf{x}$ in the current state. The relation between the volume in the reference state and the volume in the current state gives

$$d\mathbf{x} \cdot \mathbf{n}da = \det(\mathbf{F})d\mathbf{X} \cdot \mathbf{N}dA \, ,$$

or

$$dX_K F_{iK} n_i da = \det(\mathbf{F})dX_K N_K dA \; .$$

This relation holds for arbitrary $d\mathbf{X}$, so that

$$F_{iK} n_i da = \det(\mathbf{F}) N_K dA \, ,$$

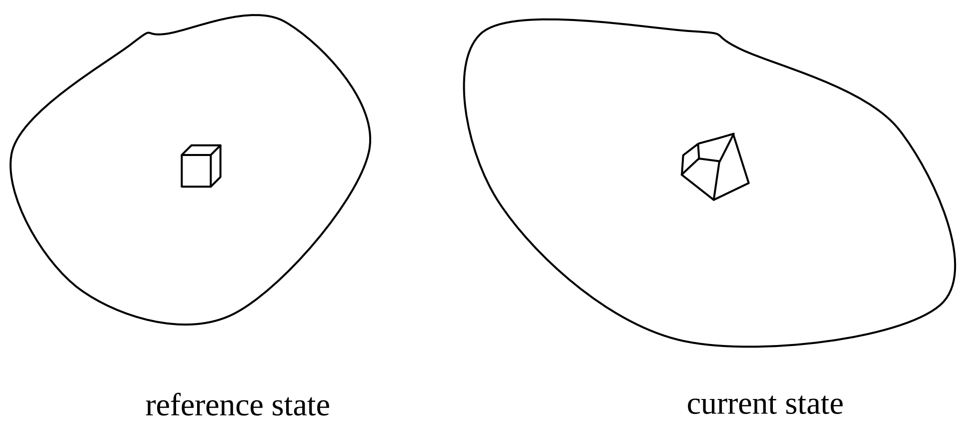
or, in vector form,

$$\mathbf{F}^T d\mathbf{a} = \det(\mathbf{F}) d\mathbf{A} \; .$$

Exercise. Consider the field of deformation in the previous exercise. When the body is in the reference state, a material plane intersects the three coordinate axes at 1, 2, and 3. Determine the unit vector of this material plane in the current state. A part of the plane is of area 2 when the body is in the reference state. Determine the area of this part when the body is in the current state.

Conservation of mass. When a body is in the reference state, a material particle occupies a place with coordinate $\mathbf{X}$, and consider a material element of volume around the particle. When the body is in a current state at t, the same material element deforms to some other shape. Let ρ be the nominal density of mass, namely,

$$\rho = \frac{\text{mass of the material element in current state}}{\text{volume of the material element in reference state}}.$$



During deformation, we assume that the material element does not gain or lose mass, so that the nominal density of mass, ρ, is time-independent. If the body in the reference state is inhomogeneous, the nominal density of mass in general varies from one material particle to another. Combining these two considerations, we write the nominal density of mass as a function of material particle:

$$\rho = \rho(\mathbf{X}).$$

This function is given as an input to our theory. The conservation of mass requires that the nominal density of mass is independent of time.

Consider a material element of volume around material particle $\mathbf{X}$. When the body in the reference state, the volume of the element is $dV(\mathbf{X})$. When the body is in the current state at time t, the material element may deform to some other volume. The mass of the material element of volume at all time is $\rho(\mathbf{X})dV(\mathbf{X})$. Consequently, at all time the mass of any part of the body is

$$\int \rho(\mathbf{X})dV.$$

The integral extends over the volume of the part in the reference state. Thus, the domain of integration remains fixed, independent of time, even though the body deforms.

Linear momentum. Consider a material element of volume around the material particle $\mathbf{X}$. When the body is in the reference state, the volume of the element is $dV(\mathbf{X})$. As the body deforms, the mass of the element remains unchanged, and is $\rho(\mathbf{X})dV(\mathbf{X})$ at all time. By definition, the linear momentum is the velocity times mass. Thus, in the current state, the material element has the linear momentum

$$\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}\rho(\mathbf{X})dV(\mathbf{X}).$$

The linear momentum of any part of the body is

$$\int \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}\rho(\mathbf{X})dV(\mathbf{X}).$$

The integral extends over the volume of the part in the reference state. The rate of change of the linear momentum of the part is

$$\frac{d}{dt}\int \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}\rho(\mathbf{X})dV(\mathbf{X}) = \int \frac{\partial^2 \mathbf{x}(\mathbf{X},t)}{\partial t^2}\rho(\mathbf{X})dV(\mathbf{X}).$$

The integrals extend over the volume of the part in the reference state.

Body force. Consider a material element of volume around material particle $\mathbf{X}$. When the body in the reference state, the volume of the element is $dV(\mathbf{X})$. When the body is in the current state at time *t*, the force acting on the element is denoted by $\mathbf{B}(\mathbf{X},t)dV(\mathbf{X})$, namely,

$$\mathbf{B}(\mathbf{X},t) = \frac{\text{force in current state}}{\text{volume in reference state}}.$$

The force $\mathbf{B}(\mathbf{X},t)dV(\mathbf{X})$ is called the *body force*, and the vector $\mathbf{B}(\mathbf{X},t)$ the nominal density of the body force. The body force is applied by an agent external to the body.

Exercise. Given a field of deformation:

$$x_1 = X_1 + X_2 \sin\gamma(t)$$
$$x_2 = X_2 \cos\gamma(t)$$
$$x_3 = X_3$$

The body is in a gravitational field, and is of nominal density of 1000kg/m^3. The gravitational field is pointing down along the x_2 axis. Calculate the nominal density of the body force due to gravitation in the current state at time *t*.

Surface force. Consider a material element of area at material particle

$\mathbf{X}$. When the body is in the reference state, the area of the element is $dA(\mathbf{X})$, and the unit vector normal to the element is $\mathbf{N}(\mathbf{X})$. When the body is in the current state at time t, the force acting on the element of area is denoted as $\mathbf{T}(\mathbf{X},t)dA(\mathbf{X})$, namely,

$$\mathbf{T}(\mathbf{X},t) = \frac{\text{force in current state}}{\text{area in reference state}}.$$

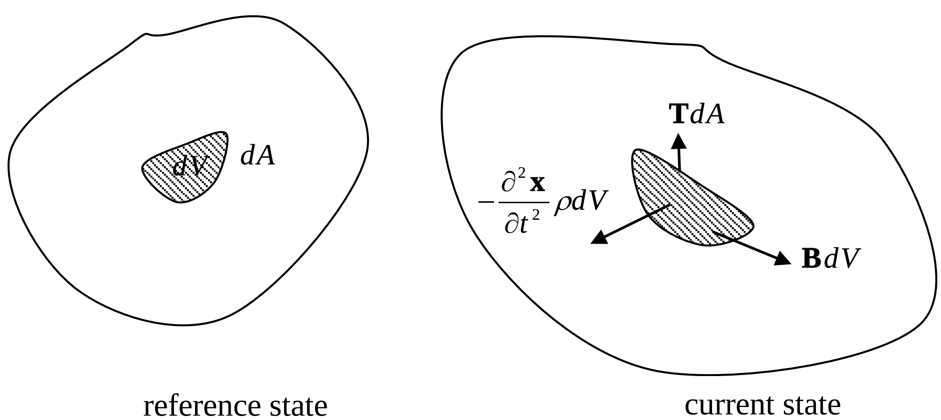
The force $\mathbf{T}(\mathbf{X},t)dA(\mathbf{X})$ is called the surface force, and the vector $\mathbf{T}(\mathbf{X},t)$ the nominal traction.

For a material element of area on the external surface, the traction may be prescribed by an agent external to the body. For a material element of area inside the body, the traction represents the force exerted by one material particle on its neighboring material particle. In this case, the traction is internal to the body, and need be determined by the theory.

Conservation of linear momentum. The law of the conservation of linear momentum requires that, for any part of a body and at any time, the force acting on the part should equal the rate of change in the linear momentum of the part:

$$\int \mathbf{T}(\mathbf{X},t)dA + \int \mathbf{B}(\mathbf{X},t)dV = \frac{d}{dt}\int \rho(\mathbf{X})\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}dV.$$

The integrals extend over the surface and the volume of any part in the reference state. That is, we regard any part of the body as a free-body diagram. We next deduce consequences of the conservation of linear momentum.



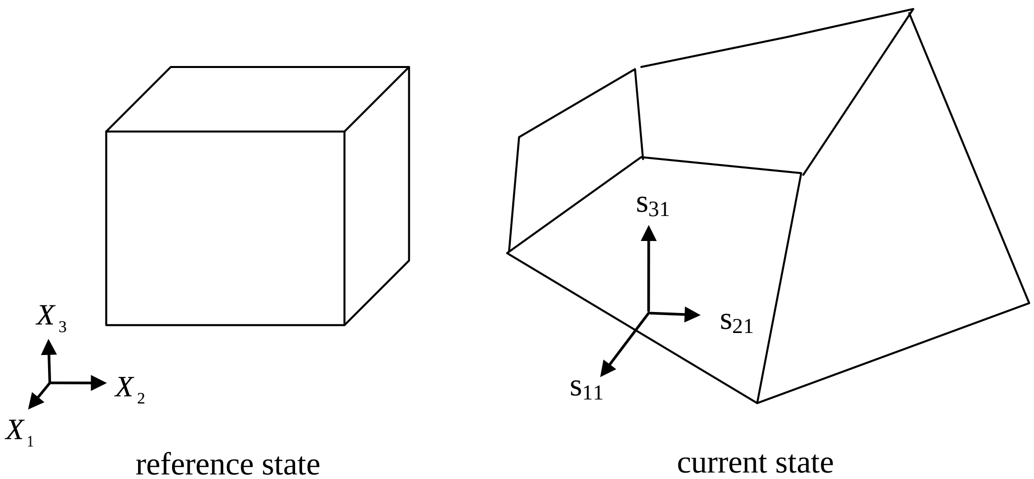
Inertial force. From the above equation, it is evident that we can regard the term

$$-\rho(\mathbf{X})\frac{\partial^2 \mathbf{x}(\mathbf{X},t)}{\partial t^2}$$

as a special type of the body force, called the inertial force. The conservation of linear momentum is viewed as a balance of forces on any part of the body, including the surface force, the body force, and the inertial force:

$$\int \mathbf{T}(\mathbf{X},t)dA + \int\left[\mathbf{B}(\mathbf{X},t) - \rho(\mathbf{X})\frac{\partial^2 \mathbf{x}(\mathbf{X},t)}{\partial t^2}\right]dV = 0.$$

Stress. For a bar pulled by an axial force, the nominal stress is defined as the axial force in the current state divided by the cross-sectional area of the bar in the reference state. We now generalize this definition to three dimensions.



Consider a small part of the body around a material particle $\mathbf{X}$. When the body is in the reference state, the small part is a rectangular block with faces parallel to the coordinate planes. When the body is in the current state at time *t*, the block deforms to some other shape. Now consider one face of the block. When the body is in the reference state, the face is a flat plane normal to the axis X_K. When the body is in the current state at time *t*, the face deforms into a curved surface. In the current state, acting on this face is a force. Denote by $s_{iK}(\mathbf{X},t)$ the component *i* of the force acting on this face in the current state divided by the area of the face in the reference state. The nine quantities s_{iK} are components of the nominal stress, or the first Piola-Kirchhoff stress.

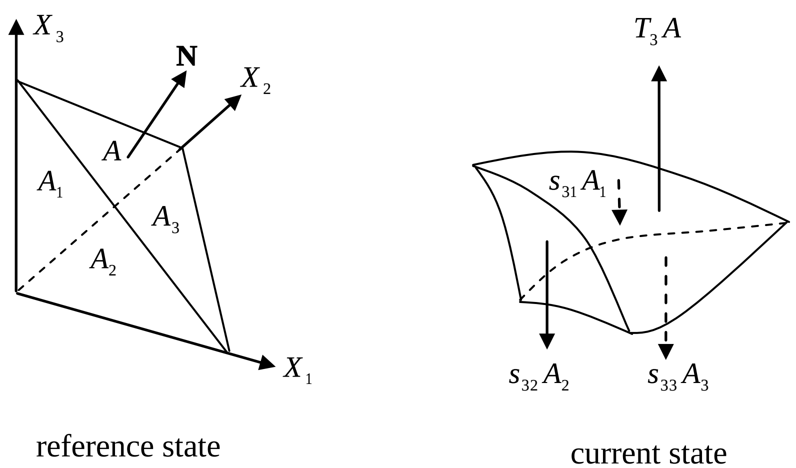
We adopt the following sign convention. When the outward normal vector of the area points in the positive direction of axis X_K, we take s_{iK} to be positive if the component *i* of the force points in the positive direction of axis x_i.

When the outward normal vector of the area points in the negative direction of the axis X_K, we take s_{iK} to be positive if the component *i* of the force points in the negative direction of axis x_i.

Stress-traction relation. Once the state of stress of a material particle is specified by s_{iK}, we know traction on all six faces of the block around the particle. This information is sufficient for us to calculate the traction at the material particle on a plane of any direction.

Consider a material tetrahedron around a material particle **X**. When the body is in the reference state, the four faces of the tetrahedron are triangles: the three triangles on the coordinate planes, and one triangle on the plane normal to the unit vector **N**. Let the areas of the three triangles on the coordinate planes be A_K, and the area of the triangle normal to **N** be *A*. The geometry dictates that

$$A_K = N_K A .$$



When the body is in the current state, the tetrahedron deforms to a shape of four curved faces. Regard this deformed tetrahedron in the current state as a free-body diagram. Now apply the law of the conservation of momentum to this deformed tetrahedron. In the current state, acting on each of the four faces is a surface force. (For clarity, only the surface forces in direction 3 are indicated in the figure.) As the volume of the tetrahedron decreases, the ratio of area over volume becomes large, so that the surface forces prevail over the body force and the change in the linear momentum. Consequently, the surface forces on the four faces of the tetrahedron must balance, giving

$$s_{i1}A_1 + s_{i2}A_2 + s_{i3}A_3 = T_i A .$$

A combination of the above two equations gives

$$s_{i1}N_1 + s_{i2}N_2 + s_{i3}N_3 = T_i .$$

We adopt the convention of summing over repeated indices, so that the above equation is abbreviated to

$$s_{iK}N_K = T_i .$$

Consequently, the nominal stress is a linear operator that maps the unit vector normal to a material plane in the reference state to the traction acting on the material plane in the current state.

Exercise. Consider a material tetrahedron in a body. When the body is in the reference state, three faces of the tetrahedron are on the coordinate planes, and the fourth face of tetrahedron intersects the three coordinate axes at 1, 2, and 3. In the current state, the tetrahedron deforms to some other shape, and the forces acting on all faces are in the direction of X_1, with the forces on faces A_1, A_2, A_3 being 4, 5, 6, respectively. Calculate the force on face A. Calculate the nominal tractions on the four faces.

Conservation of linear momentum in differential form. Consider any part of the body. The force due to traction on the surface of the part is $\int T_i dA$. This integral over the surface of the part can be converted to an integral over the volume of the part:

$$\int T_i dA = \int s_{iK}N_K dA = \int \frac{\partial s_{iK}}{\partial X_K} dV .$$

The first equality comes from the stress-traction relation, and the second equality invokes the divergence theorem.

Replacing the integral over the surface with the integral over the volume, the law of the conservation of linear momentum becomes that

$$\int \left[\frac{\partial s_{iK}(\mathbf{X},t)}{\partial X_K} + B_i(\mathbf{X},t) - \rho(\mathbf{X}) \frac{\partial^2 x_i(\mathbf{X},t)}{\partial t^2} \right] dV = 0 .$$

This equation holds for arbitrary part of the body, so that the integrand must vanish, giving

$$\frac{\partial s_{iK}(\mathbf{X},t)}{\partial X_K} + B_i(\mathbf{X},t) = \rho(\mathbf{X}) \frac{\partial^2 x_i(\mathbf{X},t)}{\partial t^2} .$$

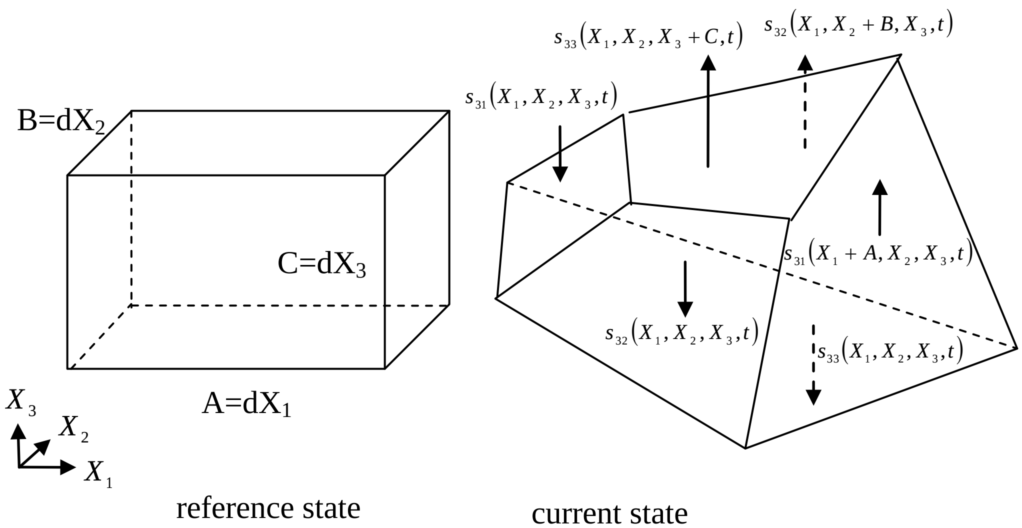
This equation expresses the conservation of linear momentum in a differential form. The equation is linear in the field of stress and the field of deformation.

If the above derivation is too mathematical for your taste, here is an alternative derivation. Consider a small block in a body. In the reference state, the block is rectangular, with faces parallel to the coordinate planes, and of sides A, B, C. In the current state, the block deforms to some other shape. In the free-body diagram of the block in the current state, we should include surface forces, body forces, and inertial forces. (For clarity, in the figure only surface forces in direction 3 on the six faces of the block are indicated.) Conserving linear momentum of this free-body diagram, we obtain that

$$\begin{aligned} & BCs_{i1}(X_1 + A, X_2, X_3, t) - BCs_{i1}(X_1, X_2, X_3, t) \\ & + CAs_{i2}(X_1, X_2 + B, X_3, t) - CAs_{i2}(X_1, X_2, X_3, t) \\ & + ABs_{i3}(X_1, X_2, X_3 + C, t) - ABs_{i3}(X_1, X_2, X_3, t) \\ & + ABCB_i(\mathbf{X}, t) \\ & = ABC\rho(\mathbf{X}) \frac{\partial^2 x_i(\mathbf{X}, t)}{\partial t^2} \end{aligned}$$

Dividing this equation by the volume of the block, *ABC*, we obtain that

$$\frac{\partial s_{iK}(\mathbf{X}, t)}{\partial X_K} + B_i(\mathbf{X}, t) = \rho(\mathbf{X}) \frac{\partial^2 x_i(\mathbf{X}, t)}{\partial t^2}.$$



Angular momentum. Let $dV(\mathbf{X})$ be a material element of volume around the material particle **X**. In the current state at time *t*, the material particle moves to the place of position vector $\mathbf{x}(\mathbf{X}, t)$, the element of volume changes to some other shape, and the linear moment of the element is $[\partial \mathbf{x}(\mathbf{X}, t) / \partial t]\rho(\mathbf{X}) dV(\mathbf{X})$. By definition, the angular momentum is the cross product of the position vector and the linear angular momentum:

$$\mathbf{x}(\mathbf{X},t)\times\frac{\partial\mathbf{x}(\mathbf{X},t)}{\partial t}\rho(\mathbf{X})dV(\mathbf{X}).$$

The linear angular momentum of any part of the body is

$$\int\mathbf{x}(\mathbf{X},t)\times\frac{\partial\mathbf{x}(\mathbf{X},t)}{\partial t}\rho(\mathbf{X})dV .$$

The integral extends over the volume of the part in the reference state.

Exercise. Show that the rate of the angular momentum of a part of the body is

$$\int\mathbf{x}(\mathbf{X},t)\times\frac{\partial^2\mathbf{x}(\mathbf{X},t)}{\partial t^2}\rho(\mathbf{X})dV .$$

The integral extends over the volume of the part in the reference state.

Conservation of angular momentum. The law of the conservation of angular momentum requires that, for any part of a body and at any time, the moment acting on the part should equal the rate of change in the angular momentum, namely,

$$\int\mathbf{x}(\mathbf{X},t)\times\mathbf{T}(\mathbf{X},t)dA+\int\mathbf{x}(\mathbf{X},t)\times\mathbf{B}(\mathbf{X},t)dV=\frac{d}{dt}\int\mathbf{x}(\mathbf{X},t)\times\frac{\partial\mathbf{x}(\mathbf{X},t)}{\partial t}\rho(\mathbf{X})dV .$$

The first term can be converted into a volume integral as follows:

$$\int\varepsilon_{ijp}x_jT_pdA=\int\varepsilon_{ijp}x_js_{pK}N_KdA=\int\varepsilon_{ijp}\frac{\partial(x_js_{pK})}{\partial X_K}dV$$

$$=\int\left[\varepsilon_{ijp}s_{pK}\frac{\partial x_j}{\partial X_K}+\varepsilon_{ijp}x_j\frac{\partial s_{pK}}{\partial X_K}\right]dV$$

where ε_{ijk} is the symbol for permutation.

Combining the above two equations, and using the equation for the conservation of linear momentum, we obtain that

$$\int\varepsilon_{ijp}s_{pK}\frac{\partial x_j}{\partial X_K}dV=0 .$$

This equation holds for arbitrary part of the body, so that the integrand must vanish:

$$\varepsilon_{ijp}s_{pK}F_{jK}=0 .$$

Recalling the definition of the permutation symbol ε_{ijp}, we find that the above equation is equivalent to

$$s_{pK}F_{jK}=s_{jK}F_{pK} .$$

We can also rewrite this equation in terms of matrices, namely,

$$\mathbf{sF}^T = \mathbf{Fs}^T .$$

That is, the law of the conservation of angular momentum requires that the product $\mathbf{sF}^T$ be a symmetric tensor. In general, neither the deformation gradient $\mathbf{F}$, nor the nominal stress $\mathbf{s}$, is a symmetric tensor.

An alternative derivation of the above equation goes as follows. Consider a small block in a body. In the reference state, the block is rectangular, with faces parallel to the coordinate planes, and of sides A, B, C. In the current state, the block deforms to some other shape. Consider the free-body diagram of the block in the current state: a pair of forces $s_{i1}BC$ act on the pair of the material elements of area BC, a pair of forces $s_{i2}CA$ act on the pair of the material elements of area CA, and a pair of forces $s_{i3}AB$ act on the pair of the material elements of area AB.

In the current state, in the coordinate plane (i, j), we balance the moments of the forces acting on the block:

$$\begin{aligned}
&BCs_{i1}\left[x_j\left(X_1 + A, X_2, X_3, t\right) - x_j\left(X_1, X_2, X_3, t\right)\right] \\
&+ CAs_{i2}\left[x_j\left(X_1, X_2 + B, X_3, t\right) - x_j\left(X_1, X_2, X_3, t\right)\right] \\
&+ ABs_{i3}\left[x_j\left(X_1, X_2, X_3 + C, t\right) - x_j\left(X_1, X_2, X_3, t\right)\right] \\
&= BCs_{j1}\left[x_i\left(X_1 + A, X_2, X_3, t\right) - x_i\left(X_1, X_2, X_3, t\right)\right] \\
&+ CAs_{j2}\left[x_i\left(X_1, X_2 + B, X_3, t\right) - x_i\left(X_1, X_2, X_3, t\right)\right] \\
&+ ABs_{j3}\left[x_i\left(X_1, X_2, X_3 + C, t\right) - x_i\left(X_1, X_2, X_3, t\right)\right]
\end{aligned}$$

Divide the equation by the volume ABC, and we obtain that

$$s_{iK}F_{jK} = s_{jK}F_{iK} .$$

Work. From a time t and to a slightly later time $t + \delta t$, a material particle $\mathbf{X}$ moves by a small displacement:

$$\delta \mathbf{x} = \mathbf{x}\left(\mathbf{X}, t + \delta t\right) - \mathbf{x}\left(\mathbf{X}, t\right).$$

Associated with this field of small displacement, the surface force, the body force, and the inertial force together do work to a part of the body:

$$\text{work} = \int T_i \delta x_i dA + \int \left(B_i - \rho \frac{\partial^2 x_i}{\partial t^2} \right) \delta x_i dV .$$

The integrals extend over the surface and the volume of the part of the body.

We can express the work in terms of the stress. Associated with the small displacement δx_i, the deformation gradient of a material particle $\mathbf{X}$ changes by

$$\delta F_{iK} = F_{iK}\left(\mathbf{X}, t + \delta t\right) - F_{iK}\left(\mathbf{X}, t\right) = \frac{\partial \delta x_i}{\partial X_K} .$$

Recall that the conservation of linear momentum results in two equations:

$$T_i = s_{iK} N_K ,$$

$$\frac{\partial s_{iK}}{\partial X_K} + B_i = \rho \frac{\partial^2 x_i}{\partial t^2} .$$

Insert the two equations into the expression for work, and apply the divergence theorem

$$\int s_{iK} N_K \delta x_i dA = \int \frac{\partial (s_{iK} \delta x_i)}{\partial X_K} dV .$$

We reduce the work done by all the forces to the following expression:

$$\text{work} = \int s_{iK} \delta F_{iK} dV .$$

This expression holds for any part of the body. Consequently, associated with a field of small displacement δx_i, all the forces together do this amount work per unit volume:

$$\frac{\text{work in the current state}}{\text{volume in the reference state}} = s_{iK} \delta F_{iK} .$$

That is, the nominal stress is work-conjugate to the deformation gradient.

An alternative derivation of the above equation goes as follows. Consider a small block of material in a body. In the reference state, the block is rectangular, of lengths A, B, C. In the current state, the block deforms to some other shape. Consider the free-body diagram of the block in the current state: a pair of forces $s_{i1} BC$ act on the pair of the material elements of area BC, a pair of forces $s_{i2} CA$ act on the pair of the material elements of area CA, and a pair of forces $s_{i3} AB$ act on the pair of the material elements of area AB. Between time t and $t + \delta t$, the six faces move a little bit, and the forces on the six faces do work

$$\begin{aligned}
& BCs_{i1}\left[x_i(X_1 + A, X_2, X_3, t + \delta t) - x_i(X_1 + A, X_2, X_3, t)\right] \\
& - BCs_{i1}\left[x_i(X_1, X_2, X_3, t + \delta t) - x_i(X_1, X_2, X_3, t)\right] \\
& + CAs_{i2}\left[x_i(X_1, X_2 + B, X_3, t + \delta t) - x_i(X_1, X_2 + B, X_3, t)\right] \\
& - CAs_{i2}\left[x_i(X_1, X_2, X_3, t + \delta t) - x_i(X_1, X_2, X_3, t)\right] \\
& + ABs_{i3}\left[x_i(X_1, X_2, X_3 + C, t + \delta t) - x_i(X_1, X_2, X_3 + C, t)\right] \\
& - ABs_{i3}\left[x_i(X_1, X_2, X_3, t + \delta t) - x_i(X_1, X_2, X_3, t)\right]
\end{aligned}$$

This expression is the same as

$$ABCs_{iK} \delta F_{iK} .$$

Consequently, the work done by the forces on the block in the current state divided by volume of the block in the reference state is

$$s_{iK}\delta F_{iK} .$$

The above result generalizes the one dimensional result for a bar pulled by a force:

$$\frac{\text{work in the current state}}{\text{volume in the reference state}} = s\delta\lambda .$$

To appreciate the general result, it might be instructive for you to review how this one-dimensional result arises from elementary considerations (http://imechanica.org/node/5065).

Helmholtz free energy. The condition of thermodynamic equilibrium for isothermal deformation. Consider an elastic body subject to an external force. We will consider isothermal deformation—that is, the body is in thermal equilibrium with a heat reservoir, held at a fixed temperature. The condition for the body in thermodynamic equilibrium with the external force can be stated as follows (http://imechanica.org/node/4878).

> *When an elastic body is in a state of thermodynamic equilibrium, the work done by the external force through any arbitrary small displacement superimposed onto the state equals the change in the Helmholtz free energy of the body.*

By definition, the Helmholtz free energy is the internal energy minus the temperature times the entropy. For an inorganic elastic body, the work done by the external force mostly causes the change in the internal energy by stretching and distorting atomic bonds. For an elastomer, however, the work done by the external force nearly does not vary the internal energy, but mostly changes the entropy. That is, the elastic deformation of an elastomer is almost entirely entropic. When a rubber band is subject to no external force, the polymer chains in the rubber band are coiled and rapidly flip among a large number of configurations. When the rubber band is pulled by an external force, the polymer chains become less coiled and flip among a smaller number of configurations. That is, the stretching reduces the entropy of the rubber band. For the heat reservoir to keep the temperature of the rubber band constant, energy flows out the rubber band and into the reservoir. You can sense the reduction in the entropy of the rubber band by pulling it rapidly, and feel the rise of the temperature with your lips.

Let W be the nominal density of the Helmholtz free energy. Consider a material element of volume. In the reference state, the element is a block of unit volume. In the current state, the block deforms to some other shape, and the faces of the block are subject to the stress s_{iK}. In the current state, for the material to be in thermodynamic equilibrium with the stress, the change in free energy must equal the work done by the stress:

$$\delta W = s_{iK} \delta F_{iK} .$$

This condition of equilibrium holds for arbitrary and independent variations of all components of the tensor $\mathbf{F}$.

Stress-strain relations. **Equations of state**. As a material model, we assume that the nominal density of the Helmholtz free energy is a function of the deformation gradient:

$$W = W(\mathbf{F}).$$

The temperature is held as a constant, and will not be included as a variable.

When the deformation gradient changes by $\delta \mathbf{F}$, we can expand the change in the free energy into the Taylor series:

$$\delta W = \frac{\partial W(\mathbf{F})}{\partial F_{iK}} \delta F_{iK} .$$

Here we have retained only the sum of the first-order terms of $\delta \mathbf{F}$.

A comparison of the two expressions of δW gives that

$$\left[s_{iK} - \frac{\partial W(\mathbf{F})}{\partial F_{iK}} \right] \delta F_{iK} = 0 .$$

This expression is a sum of nine terms. When the block of material is in equilibrium with the stresses, this expression holds for arbitrary and independent components of the tensor $\delta \mathbf{F}$. Consequently, the factor in front of each component δF_{iK} must vanish, namely,

$$s_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}} .$$

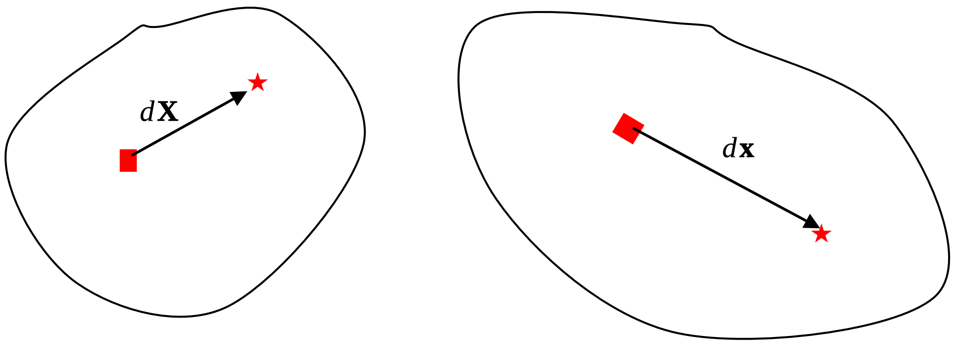
This equation gives the stress-strain relation, or the equations of state, once the function $W(\mathbf{F})$ is provided.

Rigid-body rotation does not change energy. Recall the kinematic significance of the deformation gradient $\mathbf{F}$. Consider in a body two material particles. The material element of line between the two particles is the vector $d\mathbf{X}$ when the body is in the reference state, and the same material element of line becomes the vector $d\mathbf{x}$ when the body is in the current state. By definition, the deformation gradient $\mathbf{F}$ is the tensor that maps the vector $d\mathbf{X}$ to the vector $d\mathbf{x}$, namely,

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} .$$

The deformation gradient changes both the length and the orientation of the material element of line. In the current, the square of the length is

$$(d\mathbf{x})\cdot(d\mathbf{x}) = (d\mathbf{X})^T \mathbf{F}^T \mathbf{F}(d\mathbf{X}).$$



Given an element of line, the change of the length due to deformation can be calculated if we know the Green deformation tensor:

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} ,$$

or

$$C_{KL} = F_{iK} F_{iL} .$$

The tensor **F** is in general not symmetric, and represents both stretching and rotation. The tensor **C** is symmetric, and represents stretching only.

The free energy is invariant if the material particle undergoes a rigid-body rotation in the current state. Thus, W depends on **F** only through the product $C_{KL} = F_{iK} F_{iL}$. We write

$$W(\mathbf{F}) = f(\mathbf{C}).$$

Recall that the stress is work-conjugate to the deformation gradient:

$$s_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}} .$$

Using the chain rule in calculus, we write

$$s_{iK} = \frac{\partial f(\mathbf{C})}{\partial C_{MN}} \frac{\partial C_{MN}}{\partial F_{iK}} ,$$

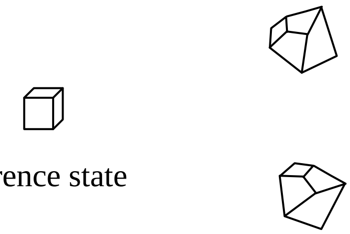
or

$$s_{iK} = 2\frac{\partial f(\mathbf{C})}{\partial C_{LK}} F_{iL} .$$

This equation obeys the equation resulting from the law of the conservation of angular momentum:

$$s_{iK}F_{jK} = s_{jK}F_{iK}.$$

Consequently, if we write the free energy as a function of the Green deformation tensor **C**, the resulting theory conserves angular momentum.



Initial and boundary value problems in elasticity. Let us summarize the above developments and state the initial boundary value problem to be solved. A body is represented by a field of material particles. Each material particle is named by its coordinate **X** when the body is in a reference state. In the current state at time *t*, the material particle occupies the place with coordinate **x**. The object of the theory of elasticity is to determine the function $\mathbf{x}(\mathbf{X},t)$.

The body is prescribed with a field of mass density, $\rho(\mathbf{X})$. A material model is specified by a free-energy function $W(\mathbf{F})$, which depends on **F** through the product $\mathbf{F}^T\mathbf{F}$. The body is subject to a field of body force $\mathbf{B}(\mathbf{X},t)$. The governing equations are

$$F_{iK} = \frac{\partial x_i(\mathbf{X},t)}{\partial X_K},$$

$$\frac{\partial s_{iK}(\mathbf{X},t)}{\partial X_K} + B_i(\mathbf{X},t) = \rho(\mathbf{X})\frac{\partial^2 x_i(\mathbf{X},t)}{\partial t^2},$$

$$s_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}}.$$

In this formulation, the first two equations are linear, but the last equation is typically nonlinear. The above three equations evolve the field of deformation $\mathbf{x}(\mathbf{X},t)$ in time, subject to the following initial and boundary conditions.

Initial conditions are given by prescribing at time t_0 the places of all the particles, $\mathbf{x}(\mathbf{X},t_0)$, and the velocities of all the particles, $\mathbf{V}(\mathbf{X},t_0)$.

For every material particle on the surface of the body, we prescribe either one of the following two boundary conditions. On part of the surface of the body, S_t, the traction is prescribed, so that

$$s_{iK}(\mathbf{X},t)N_K(\mathbf{X}) = \text{prescribed}, \quad \text{for } \mathbf{X} \in S_t$$

On the other part of the surface of the body, S_u, the position is prescribed, so that

$$\mathbf{x}(\mathbf{X},t) = \text{prescribed}, \quad \text{for } \mathbf{X} \in S_u.$$

Now we have the basic equations. What do we do next? The above formulation of the boundary-value problem, in one form or another, has existed for well over a century. However, exploration of its consequences remains active to this day. Representative activities include

- Model a specific elastic material by constructing a function $W(\mathbf{F})$, by a combination of microcosmic modeling and experimental testing.
- Model a specific phenomenon of elastic deformation by formulating a boundary-value problem.
- Analyze such a boundary-value problem by analytic techniques, such as dimensional analysis and linear perturbation.
- Analyze such a boundary-value problem by numerical methods, such as using commercial finite element package.

Of course, you can also play another kind of game: you can use the similar approach to formulate models for phenomena other than the deformation of an elastic body.

Isotropic material. To specify an elastic material model, we need to specify the nominal density of Helmholtz free energy as a function of the Green deformation tensor, $\mathbf{C}$. That is, we need to specify the function

$$W = f(\mathbf{C}).$$

The tensor $\mathbf{C}$ is positive-definite and symmetric. In three dimensions, this tensor has 6 independent components. Thus, to specify an elastic material model, we need to specify the free energy as a function of the 6 variables. For a given material, such a function is specified by a combination of experimental measurements and theoretical considerations. Trade off is made between the amount of effort and the need for accuracy.

For an isotropic material, the free-energy density is unchanged when the coordinates rotate in the reference state. When the coordinates rotate, however, the individual components of the tensor $\mathbf{C}$ will change. How do we write the function $W(\mathbf{C})$ for an isotropic material?

For any symmetric second-rank tensor $\mathbf{C}$, the six components of the tensor form three scalars:

$$\alpha = C_{KK},$$
$$\beta = C_{KL}C_{KL},$$
$$\gamma = C_{KL}C_{LJ}C_{JK}.$$

These scalars are formed by combining the individual components of the tensor in ways that make all indices dummy. The three scalars remain unchanged when the coordinates rotate, and are known as the invariants of the tensor **C**.

To specify model of an isotropic elastic material, we need to prescribe the free-energy density as a function of the three invariants of the Green deformation tensor:

$$W = f(\alpha,\beta,\gamma).$$

Once this function is specified, we can derive the stress by using the chain rule:

$$s_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}} = \frac{\partial f}{\partial \alpha}\frac{\partial \alpha}{\partial F_{iK}} + \frac{\partial f}{\partial \beta}\frac{\partial \beta}{\partial F_{iK}} + \frac{\partial f}{\partial \gamma}\frac{\partial \gamma}{\partial F_{iK}}.$$

Derivatives of this kind are recorded in many textbooks on continuum mechanics.

Exercise. Invariants of a symmetric second-rank tensor can be written in many forms. In many textbooks, the invariants are introduced by using the characteristic equations. Relate invariants introduced by the characteristic equations and those written above.

Incompressible material. Subject to external forces, elastomers can undergo large change in shape, but very small change in volume. A commonly used idealization is to assume that such materials are incompressible, namely,

$$\det \mathbf{F} = 1.$$

In arriving at the relation $s_{iK} = \partial W(\mathbf{F})/\partial F_{iK}$, we have assumed that each component of F_{iK} can vary independently. However, the condition of incompressibility places a constraint among the components. To enforce this constraint, we replace the free-energy function $W(\mathbf{F})$ by a function

$$W(\mathbf{F}) - \Pi(\det \mathbf{F} - 1),$$

with Π as a Lagrange multiplier. We then allow each component of F_{iK} to vary independently, and obtain the stress from

$$s_{iK} = \frac{\partial}{\partial F_{iK}}\left[W(\mathbf{F}) - \Pi(\det \mathbf{F} - 1)\right].$$

Recall an identity in the calculus of matrix:

$$\frac{\partial \det \mathbf{F}}{\partial F_{iK}} = H_{iK} \det \mathbf{F} ,$$

where $\mathbf{H}$ is defined by $H_{iK}F_{iL} = \delta_{KL}$ and $H_{iK}F_{jK} = \delta_{ij}$. For a proof see p. 41 of Holzapfel.

Thus, for an incompressible material, the stress relates to the deformation gradient as

$$s_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}} - \Pi H_{iK} .$$

This equation, along with the constraint $\det \mathbf{F} = 1$, specifies a material model for incompressible elastic materials. The Lagrange multiplier Π is not a material parameter. Rather, Π is to be solved as a field for a given boundary-value problem.

Exercise. In the notes on special cases of finite deformation (http://imechanica.org/node/5065), we have also enforced incompressibility in terms of principal stretches. However, we did not use the method of Lagrange multiplier. Now try to enforce incompressibility in terms of principal stretches by using a Lagrange multiplier.

Neo-Hookean material. For a neo-Hookean material, the free-energy function takes the form

$$W(\mathbf{F}) = \frac{\mu}{2}\left(F_{iK}F_{iK} - 3\right).$$

Note that $C_{KK} = F_{iK}F_{iK}$ is an invariant of the Green deformation tensor $\mathbf{C}$. The material is also taken to be incompressible, namely,

$$\det \mathbf{F} = 1 .$$

The stress relates to the deformation gradient as

$$s_{iK} = \mu F_{iK} - \Pi H_{iK} .$$

Exercise. In the notes on special cases of finite deformation (http://imechanica.org/node/5065), several forms of free energy function are given in terms of the principal stretches. They can be rewritten in terms of deformation gradient, following the same procedure outlined above. Go through this procedure for the Gent model.

Weak statement of the conservation of momentum. The conservation of linear momentum results in two equations:

$$T_i = s_{iK} N_K ,$$

$$\frac{\partial s_{iK}}{\partial X_K} + B_i = \rho \frac{\partial^2 x_i}{\partial t^2} .$$

This pair of equations may be called the *strong statement* of the conservation of momentum.

Denote an arbitrary field by

$$\Delta_i = \Delta_i(\mathbf{X}).$$

Consider any part of the body. Multiplying $\Delta_i(\mathbf{X})$ to the two equations of the strong statement, integrating over the surface of the part and the volume of the part, respectively, and then adding the two, we obtain that

$$\int s_{iK} \frac{\partial \Delta_i}{\partial X_K} dV = \int T_i \Delta_i dA + \int \left(B_i - \rho \frac{\partial^2 x_i}{\partial t^2} \right) \Delta_i dV .$$

The integrals extend over the volume and the surface of the part.

In reaching the above equation, we have used the divergence theorem. The conservation of linear momentum implies that the above equation holds for any field $\Delta_i(\mathbf{X})$. This statement is known as the *weak statement* of the conservation of linear momentum. The field $\Delta_i(\mathbf{X})$ is called a *test function*.

Exercise. Start with the weak statement of the conservation of linear momentum, and show that the weak statement implies the strong statement.

Virtual work. We now paraphrase the weak statement of the conservation of linear momentum. The test function $\Delta_i(\mathbf{X})$ is sometimes called *virtual displacement*. From the above development, the test function needs not to have any physical interpretation. Indeed, $\Delta_i(\mathbf{X})$ does not need to be small, and nor does it have the dimension of displacement. That is, $\Delta_i(\mathbf{X})$ is an arbitrary field, and is totally unrelated to displacement, or any physical quantity.

The quantity

$$\int T_i \Delta_i dA + \int \left(B_i - \rho \frac{\partial^2 x_i}{\partial t^2} \right) \Delta_i dV$$

is called the *virtual work*. The integral extends over any part of a body. The virtual work is done by the actual forces—the surface force, the body force and the inertial force—in the current state, through the test function (i.e., the virtual displacement).

On revisiting the discussion of the (actual) work, we can readily confirm that

$$\frac{\text{virtual work in the current state}}{\text{volume in the reference state}} = s_{iK}\frac{\partial \Delta_i(\mathbf{X})}{\partial X_K}.$$

Here s_{iK} is the actual stress in the current state. Following the fake idea, we might as well call the quantity $\partial \Delta_i(\mathbf{X})/\partial X_K$ the *virtual displacement gradient*.

In terms of these virtual quantities, the weak statement of the linear momentum,

$$\int s_{iK}\frac{\partial \Delta_i}{\partial X_K}dV = \int T_i \Delta_i dA + \int\left(B_i - \rho\frac{\partial^2 x_i}{\partial t^2}\right)\Delta_i dV ,$$

acquires a new name: the principle of virtual work. The left-hand side is called the internal virtual work, and the right-hand side is called the external virtual work.

Exercise. The weak statement is the basis for the finite element method (http://imechanica.org/node/324). In that context, the volume integrals extend over the entire body, and the surface integral extends over the part of the surface of the body on which traction is prescribed. The test function $\Delta_i(\mathbf{X})$ is set to vanish on part of the surface where displacement is prescribed.

An elastic material is modeled by prescribing the nominal density of the Helmholtz free energy as a function of the deformation gradient, $W(\mathbf{F})$. To ensure that the free energy is invariant when the body undergoes a rigid-body rotation, we require that W depend on $\mathbf{F}$ only through the Green deformation tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. Recall that

$$s_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}}.$$

Sketch the finite element method by using the above equation.

Alternative representations of the same ingredients in continuum mechanics. In the following we represent various ingredients of the theory in alternative forms. Of course, the new forms add no new substance. But these alternative forms appear in the literature so frequently that you should know some of them. Besides, alternative representations of an ingredient may shed additional light.

Lagrange strain. Once again consider a material element of line in a body. When the body is in the reference state, the element is between two nearby material particles, $\mathbf{X}$ and $\mathbf{X} + d\mathbf{X}$, the element is represented by the vector $d\mathbf{X}$, and the length of the element, dL, is calculated from

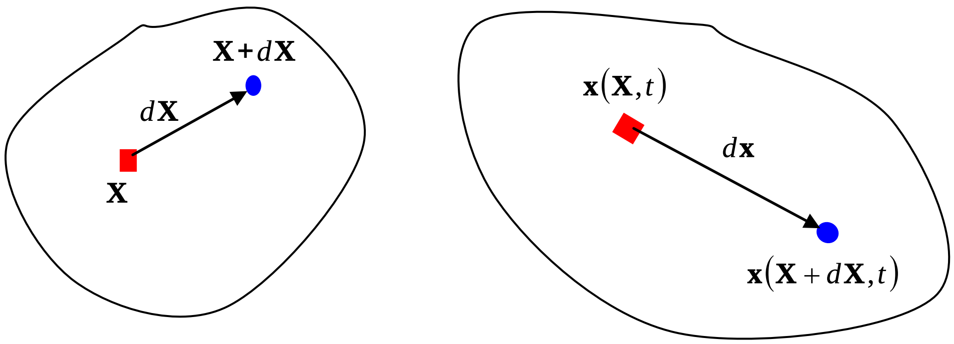
$$dL^2 = dX_K dX_K .$$

When the body is in the current state, the material particle $\mathbf{X}$ moves to the place $\mathbf{x}(\mathbf{X},t)$, the material particle $\mathbf{X}+d\mathbf{X}$ moves to the place, $\mathbf{x}(\mathbf{X}+d\mathbf{X},t)$, the element of line is represented by the vector $d\mathbf{x} = \mathbf{x}(\mathbf{X}+d\mathbf{X},t) - \mathbf{x}(\mathbf{X},t)$, and the length of the element, dl, is calculated from

$$dl^2 = dx_i dx_i .$$

Recall the relation $dx_i = F_{iK} dX_K$, and we write

$$dl^2 = F_{iK} F_{iL} dX_K dX_L .$$



reference state current state

The above expressions lead to

$$dl^2 - dL^2 = (F_{iK} F_{iL} - \delta_{KL}) dX_K dX_L .$$

Consequently, given a material element of line, $d\mathbf{X}$, the above equation gives the change in the length of the element caused by the deformation of the body.

Define the Lagrange strain as

$$E_{KL} = \frac{1}{2}(F_{iK} F_{iL} - \delta_{KL}),$$

where $\delta_{KL} = 1$ when $K = L$, and $\delta_{KL} = 0$ when $K \neq L$. The E_{KL} tensor is symmetric. The above expression can also be written as

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I}).$$

Recall the definition of the Green deformation tensor $\mathbf{C} = \mathbf{F}^T \mathbf{F}$. The Lagrange strain tensor relates to the Green deformation tensor as

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}).$$

Lagrange strain of a material element of line. We can relate the general definition of the Lagrange strain to that introduced in describing a tensile bar (http://imechanica.org/node/5065). When the length of the bar increases from L to l, the Lagrange strain is defined as

$$\eta = \frac{1}{2}\left[\left(\frac{l}{L}\right)^2 - 1\right].$$

Consider a material element of line in a three dimensional body. In the reference state, the element of line is in the direction specified by a unit vector **M**. Recall the expression

$$dl^2 - dL^2 = \left(F_{iK}F_{iL} - \delta_{KL}\right)dX_K dX_L .$$

Dividing both sides by dL, we obtain that

$$\left(\frac{dl}{dL}\right)^2 - 1 = \left(F_{iK}F_{iL} - \delta_{KL}\right)dM_K dM_L$$

In the current state, the stretch of the element is $\lambda = dl/dL$. Consequently, the Lagrange strain of the element in direction **M** is given by

$$\eta = E_{KL}M_K M_L ,$$

or

$$\eta = \mathbf{M}^T\mathbf{E}\mathbf{M} .$$

Thus, once we know the tensor **E**, we can calculate the Lagrange strain η of a material element of line in any direction **M**.

Exercise. The Lagrange strain tensor can also be used to calculate the engineering shear strain. Consider two material elements of line in a body. When the body is in the reference state, the two elements are in two orthogonal directions $\mathbf{M}$ and $\mathbf{N}$. When the body is in the current state, the stretches of the two elements are $\lambda_{\mathbf{M}}$ and $\lambda_{\mathbf{N}}$, and the angle between the two elements become $\frac{\pi}{2} - \gamma$. We have defined γ as the engineering shear strain. Show that

$$\sin\gamma = \frac{2E_{KL}M_K N_L}{\lambda_{\mathbf{M}}\lambda_{\mathbf{N}}} .$$

Exercise. The Lagrange strain links finite deformation and the approximation of infinitesimal deformation. Define the displacement of the material particle **X** at time t by

$$\mathbf{u} = \mathbf{x}(\mathbf{X},t) - \mathbf{X} .$$

The displacement is a field, $\mathbf{u}(\mathbf{X},t)$. Show that the Lagrange strain is

$$E_{KL} = \frac{1}{2}\left[\frac{\partial u_K(\mathbf{X},t)}{\partial X_L} + \frac{\partial u_L(\mathbf{X},t)}{\partial X_K} + \frac{\partial u_i(\mathbf{X},t)}{\partial X_K}\frac{\partial u_i(\mathbf{X},t)}{\partial X_L} \right].$$

Thus, the Lagrange strain coincides with the strain obtained in the infinitesimal strain formulation if

$$\frac{\partial u_i(\mathbf{X},t)}{\partial X_K} << 1 .$$

This in turn requires that all components of linear strain and rotation be small. However, even when all strains and rotations are small, we still need to balance force in the deformed state, as remarked in the notes on special cases of finite deformation (http://imechanica.org/node/5065).

The second Piola-Kirchhoff stress. Define the second Piola-Kirchhoff stress, S_{KL}, by

$$\frac{\text{work in the curent state}}{\text{volume in the reference state}} = S_{KL} \delta E_{KL} .$$

This expression defines a new measure of stress, S_{KL}. Because E_{KL} is a symmetric tensor, we can set S_{KL} to be symmetric.

Recall that we have also expressed the same work by $s_{iK}\delta F_{iK}$. Equating the two expressions for work, we write

$$s_{iK}\delta F_{iK} = S_{KL}\delta E_{KL} .$$

Recall definition of the Lagrange strain,

$$E_{KL} = \frac{1}{2}\left(F_{iK}F_{iL} - \delta_{KL}\right) .$$

We obtain that

$$\delta E_{KL} = \frac{1}{2}\left(F_{iL}\delta F_{iK} + F_{iK}\delta F_{iL}\right)$$

and

$$s_{iK}\delta F_{iK} = S_{KL}F_{iL}\delta F_{iK} .$$

Here we have used the symmetry $S_{KL} = S_{LK}$. In the above equation, each side is a sum of nine terms. Each component of δF_{iK} is an arbitrary and independent variation. Consequently, the factors in front of each component of δF_{iK} must equal, giving

$$s_{iK} = S_{KL}F_{iL} ,$$

or

$$\mathbf{s} = \mathbf{F}^T\mathbf{S} .$$

This equation relates the first Piola-Kirchhoff stress to the second Piola-Kirchhoff stress.

Because both tensors, S_{KL} and E_{KL} are invariant when the body undergoes a rigid-body rotation, the two tensors are often used in formulating stress-strain relations. Thus, one writes

$$\mathbf{S} = \mathbf{f}(\mathbf{E}).$$

For an elastic material, the nominal density of the Helmholtz free energy is a function of the Lagrange strain, $W(\mathbf{E})$. The second Piola-Kirchhoff stress is work-conjugate to the Lagrange strain:

$$S_{KL} = \frac{\partial W(\mathbf{E})}{\partial E_{KL}}.$$

Exercise. Derive the differential form of the conservation of linear momentum in terms of the second Piola-Kirchhoff stress.

Exercise. For the neo-Hookean material, write the stress-strain relations in terms of the second Piola-Kirchhoff stress and the Lagrange strain.

Lagrangian vs. Eulerian formulations. The above formulation uses the material coordinate $\mathbf{X}$ and time t as independent variables, a formulation known as the Lagrangian formulation. The formulation results in initial boundary value problems that evolve in time various fields over the body in the reference state.

The function $\mathbf{x}(\mathbf{X},t)$ gives the place occupied by the material particle $\mathbf{X}$ at time t. The inverse function, $\mathbf{X}(\mathbf{x},t)$, tells us which material particle is at place $\mathbf{x}$ at time t. The formulation using the spatial coordinate $\mathbf{x}$ is known as the Eulerian formulation.

We have used the Eulerian formulation to study cavitation because we know the current configuration (http://imechanica.org/node/5065). In general, the current configuration is unknown, and we have used the Lagrangian formulation to state the general problem. In the following, we list a few results related to the Eulerian formulation.

Time derivative of a function of material particle. At time t, a material particle $\mathbf{X}$ moves to position $\mathbf{x}(\mathbf{X},t)$. The velocity of the material particle is

$$\mathbf{V} = \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}.$$

The velocity is a field, $\mathbf{V}(\mathbf{X},t)$. Here, the independent variables are the time and the coordinates of the material particles when the body is in the reference state.

We can also use $\mathbf{x}$ as the independent variable. To do so we change the variable from $\mathbf{X}$ to $\mathbf{x}$ by using the function $\mathbf{X}(\mathbf{x},t)$, and then write

$$\mathbf{v}(\mathbf{x},t) = \mathbf{V}(\mathbf{X},t).$$

Both $\mathbf{v}(\mathbf{x},t)$ and $\mathbf{V}(\mathbf{X},t)$ represent the same physical object: the velocity of material particle $\mathbf{X}$ at time t. Because of the change of variables, from $\mathbf{X}$ to $\mathbf{x}$, the two functions $\mathbf{v}(\mathbf{x},t)$ and $\mathbf{V}(\mathbf{X},t)$ are different. We indicate this difference by using different symbols, $\mathbf{v}$ and $\mathbf{V}$.

This practice, however, is not always convenient. Often we simply use the same symbol for the velocity, and then indicate the independent variables: $\mathbf{v}(\mathbf{x},t)$ and $\mathbf{v}(\mathbf{X},t)$. They represent the same physical object: the velocity of material particle $\mathbf{X}$ at time t.

Let $G(\mathbf{X},t)$ be a function of material particle and time. For example, G can be the temperature of material particle $\mathbf{X}$ at time t. The rate of change in temperature of the material particle is

$$\frac{\partial G(\mathbf{X},t)}{\partial t}.$$

This rate is known as the *material time derivative*.

We can calculate the material time derivative by an alternative approach. Change the variable from $\mathbf{X}$ to $\mathbf{x}$ by using the function $\mathbf{X}(\mathbf{x},t)$, and write

$$g(\mathbf{x},t) = G(\mathbf{X},t)$$

Using chain rule, we obtain that

$$\frac{\partial G(\mathbf{X},t)}{\partial t} = \frac{\partial g(\mathbf{x},t)}{\partial t} + \frac{\partial g(\mathbf{x},t)}{\partial x_i}\frac{\partial x_i(\mathbf{X},t)}{\partial t}.$$

Thus, we can calculate the material time derivative from

$$\frac{\partial G(\mathbf{X},t)}{\partial t} = \frac{\partial g(\mathbf{x},t)}{\partial t} + \frac{\partial g(\mathbf{x},t)}{\partial x_i} v_i(\mathbf{x},t).$$

Let us apply the idea to the acceleration of a material particle. In the Lagrangian formulation, the acceleration is

$$\mathbf{A}(\mathbf{X},t) = \frac{\partial \mathbf{V}(\mathbf{X},t)}{\partial t} = \frac{\partial^2 \mathbf{x}(\mathbf{X},t)}{\partial t^2}.$$

In the Eulerian formulation, the acceleration of a material particle is

$$a_i(\mathbf{x},t) = \frac{\partial v_i(\mathbf{x},t)}{\partial t} + \frac{\partial v_i(\mathbf{x},t)}{\partial \mathbf{x}_j} v_j(\mathbf{x},t).$$

Small displacement and its gradient. Consider a body undergoing a deformation $\mathbf{x}(\mathbf{X},t)$. When the body is in the reference state, the coordinate of a particle is $\mathbf{X}$. The material particle moves to a place $\mathbf{x}(\mathbf{X},t)$ at time *t*, and to another place $\mathbf{x}(\mathbf{X},t+\delta t)$ at time $t+\delta t$. Thus, when time goes from t to $t+\delta t$, the particle $\mathbf{X}$ moves by a small displacement:

$$\delta\mathbf{u}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t+\delta t) - \mathbf{x}(\mathbf{X},t).$$

A nearby material particle is at $\mathbf{X}+d\mathbf{X}$ when the body is in the reference state. The material particle moves to a place $\mathbf{x}(\mathbf{X}+d\mathbf{X},t)$ at time *t*, and to another place $\mathbf{x}(\mathbf{X}+d\mathbf{X},t+\delta t)$ at time $t+\delta t$. Thus, when time goes from t to $t+\delta t$, the particle $\mathbf{X}+d\mathbf{X}$ moves by another small displacement:

$$\delta\mathbf{u}(\mathbf{X}+d\mathbf{X},t) = \mathbf{x}(\mathbf{X}+d\mathbf{X},t+\delta t) - \mathbf{x}(\mathbf{X}+d\mathbf{X},t).$$

The displacement of particle $\mathbf{X}+d\mathbf{X}$ relative to that of particle $\mathbf{X}$ is

$$d\delta\mathbf{u}(\mathbf{X},t) = \delta\mathbf{u}(\mathbf{X}+d\mathbf{X},t) - \delta\mathbf{u}(\mathbf{X},t)$$

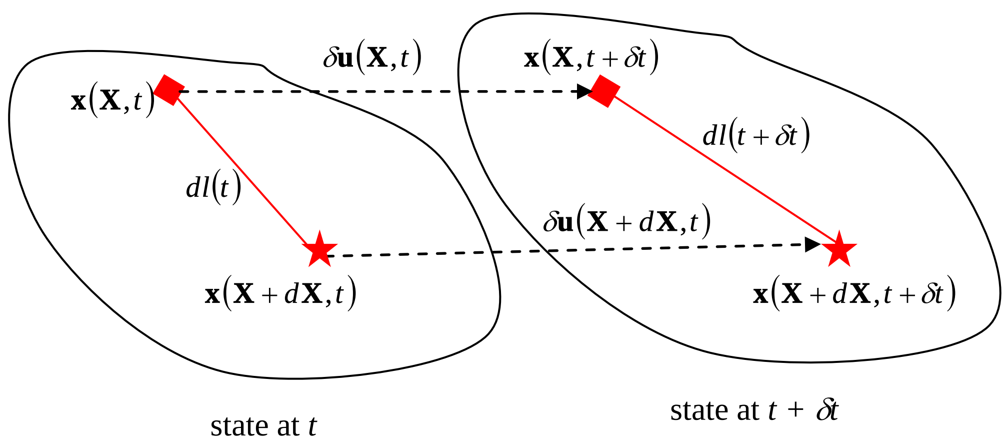
or

$$d\delta\mathbf{u}(\mathbf{X},t) = \frac{\partial \delta\mathbf{u}(\mathbf{X},t)}{\partial X_K} dX_K = \frac{\partial \delta\mathbf{u}(\mathbf{x},t)}{\partial x_i} dx_i .$$

Note that we have changed the independent variable from $\mathbf{X}$ to $\mathbf{x}$. The quantity

$$\frac{\partial \delta\mathbf{u}(\mathbf{x},t)}{\partial x_i}$$

is the gradient of the small displacement.



Updated Lagrangian strain. When the body is in the reference state, the coordinates of two nearby particles are $\mathbf{X}$ and $\mathbf{X}+d\mathbf{X}$. At time *t*, the two material particles occupy places $\mathbf{x}(\mathbf{X},t)$ and $\mathbf{x}(\mathbf{X}+d\mathbf{X},t)$, so that the vector between the two particles is

$$d\mathbf{x}(\mathbf{X},t)=\mathbf{x}(\mathbf{X}+d\mathbf{X},t)-\mathbf{x}(\mathbf{X},t).$$

In the current state, the material element of line becomes $d\mathbf{x}=\mathbf{m}dl$, where dl is the length of the element, and $\mathbf{m}$ is the unit vector in the direction of the element. The length $dl(t)$ of the material element of line is calculated from

$$dl^2 = dx_i dx_i .$$

This expression is rewritten more explicitly as

$$[dl(t)]^2 = [x_i(\mathbf{X}+d\mathbf{X},t)-x_i(\mathbf{X},t)][x_i(\mathbf{X}+d\mathbf{X},t)-x_i(\mathbf{X},t)],$$

or

$$[dl(t)]^2 = [dx_i(\mathbf{X},t)][dx_i(\mathbf{X},t)].$$

At a slightly later time, $t+\delta t$, the two material particles occupy places $\mathbf{x}(\mathbf{X},t+\delta t)$ and $\mathbf{x}(\mathbf{X}+d\mathbf{X},t+\delta t)$, so that the length $dl(t+\delta t)$ of the material element of line is calculated from

$$[dl(t+\delta t)]^2 = [x_i(\mathbf{X}+d\mathbf{X},t+\delta t)-x_i(\mathbf{X},t+\delta t)][x_i(\mathbf{X}+d\mathbf{X},t+\delta t)-x_i(\mathbf{X},t+\delta t)] ,$$

or

$$[dl(t+\delta t)]^2 = [dx_i(\mathbf{X},t+\delta t)][dx_i(\mathbf{X},t+\delta t)].$$

Consequently,

$$\begin{aligned}[dl(t+\delta t)]^2 &= [dx_i(\mathbf{X},t)+d\delta u_i(\mathbf{X},t)][dx_i(\mathbf{X},t)+d\delta u_i(\mathbf{X},t)]\\ &= dx_i(\mathbf{X},t)dx_i(\mathbf{X},t)+2d[\delta u_i(\mathbf{X},t)]dx_i(\mathbf{X},t)+d[\delta u_i(\mathbf{X},t)]d[\delta u_i(\mathbf{X},t)]\end{aligned}$$

The increment is

$$\begin{aligned}[dl(t+\delta t)]^2-[dl(t)]^2 &= 2d\delta u_i(\mathbf{X},t)dx_i(\mathbf{X},t)\\ &= 2\frac{\partial[\delta u_i(\mathbf{x},t)]}{\partial x_j}dx_j dx_i + \frac{\partial[\delta u_k(\mathbf{x},t)]}{\partial x_j}\frac{\partial[\delta u_k(\mathbf{x},t)]}{\partial x_i}dx_j dx_i\end{aligned}$$

Note that we have changed in the independent variable from $\mathbf{X}$ to $\mathbf{x}$. This expression is reminiscent of the expression for dl^2-dL^2 in the development of the Lagrange strain. Here we have used the body at time *t* as the reference state, and the body at time $t+\delta t$ as the current state. Subsequently, we will drop the second-order term in the small displacement, and write

$$[dl(t+\delta t)]^2-[dl(t)]^2 = 2\frac{\partial[\delta u_i(\mathbf{x},t)]}{\partial x_j}dx_j dx_i \ .$$

Only the symmetric part

$$\frac{1}{2}\left[\frac{\partial(\delta u_i)}{\partial x_j}+\frac{\partial(\delta u_j)}{\partial x_i}\right]$$

will affect the increment of the strain of the material element of line. This tensor generalizes the increment in the true strain, and is given the symbol:

$$\delta\varepsilon_{ij}(\mathbf{x},t)=\frac{1}{2}\left[\frac{\partial[\delta u_i(\mathbf{x},t)]}{\partial x_j}+\frac{\partial[\delta u_j(\mathbf{x},t)]}{\partial x_i}\right].$$

This tensor is known as the updated Lagrange strain.

We should have expected this result by looking at the expression for the Lagrange strain:

$$E_{KL}=\frac{1}{2}\left[\frac{\partial u_K(\mathbf{X},t)}{\partial X_L}+\frac{\partial u_L(\mathbf{X},t)}{\partial X_K}+\frac{\partial u_i(\mathbf{X},t)}{\partial X_K}\frac{\partial u_i(\mathbf{X},t)}{\partial X_L}\right].$$

When the body at time t is taken as the reference state, and the body at time $t+\delta t$ as the current state is taken as the current state, the displacement of a material particle between the two time is small, and is expressed $\delta u_i(\mathbf{x},t)$. The term of the second-order in the small displacement is negligible. With these considerations, we reduce the Lagrange strain to the updated Lagrange strain.

Increment of the length of a material element of line. Recall the definition of the true strain for a tensile bar (http://imechanica.org/node/5065). When the length of the bar changes by a small amount, from $l(t)$ to $l(t+\delta t)$, the increment of the true strain is defined as

$$\delta\varepsilon=\frac{l(t+\delta t)-l(t)}{l(t)}.$$

For a small increment of time, δt, the change in length is small, and the above expression is equivalent to

$$\delta\varepsilon=\frac{[l(t+\delta t)]^2-[l(t)]^2}{2[l(t)]^2}.$$

This definition looks similar to that of the Lagrange strain, provided we take the body at time t as the reference state, and the body at time $t+\delta t$ as the current state. The increment in the true strain is also known as the updated Lagrange strain.

Motivated by the definition of the tensile bar, we define the increment in the strain of the element line from time t to time $t+\delta t$ by

$$\delta\varepsilon=\frac{[dl(t+\delta t)]^2-[dl(t)]^2}{2[dl(t)]^2}.$$

A direct substitution gives that

$$\delta\varepsilon = \delta\varepsilon_{ij} m_i m_j .$$

Once we know the tensor $\delta\varepsilon_{ij}(\mathbf{x},t)$, we can calculate the increment of true strain in a material element in any direction.

Rate of deformation. Let $\mathbf{v}(\mathbf{x},t)$ be the field of particle velocity in the current state. The same line of reasoning shows that the tensor

$$d_{ij}(\mathbf{x},t) = \frac{1}{2}\left[\frac{\partial v_i(\mathbf{x},t)}{\partial x_j} + \frac{\partial v_j(\mathbf{x},t)}{\partial x_i}\right]$$

allows us to calculate the rate of change the length of material element of line, namely,

$$\frac{1}{dl(\mathbf{X},t)}\frac{\partial[dl(\mathbf{X},t)]}{\partial t} = d_{ij} m_i m_j .$$

The tensor d_{ij} is known as the rate of deformation.

Conservation of mass. In the Lagrangian formulation, the **nominal mass density** is defined by

$$\rho_R = \frac{\text{mass in current state}}{\text{volume in reference state}} .$$

That is, $\rho_R dV$ is the mass of a material element of volume. A subscript is added here to remind us that the volume is in the reference state.

In the Eulerian formulation, the **true mass density** is defined by

$$\rho = \frac{\text{mass in current state}}{\text{volume in current state}} .$$

That is, ρdv is the mass of a spatial element of volume.

The two definitions of density are related as

$$\rho_R dV = \rho dv ,$$

or

$$\rho_R = \rho \det \mathbf{F} .$$

The conservation of mass requires that the mass of the material element of volume be time-independent. Thus, the nominal density can only vary with material particle, $\rho_R(\mathbf{X})$, and is time-independent. By contrast, the true density is a function of both place and time, $\rho(\mathbf{x},t)$. The conservation of mass requires that

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t}+\frac{\partial}{\partial x_i}\left[\rho(\mathbf{x},t)v_i(\mathbf{x},t)\right]=0 .$$

When the material is incompressible, $\det \mathbf{F}=1$, we obtain that

$$\rho_R(\mathbf{X})=\rho(\mathbf{x},t).$$

Linear momentum. We first recall the linear momentum as an integral over material particles, and then express the linear momentum as an integral over special coordinates.

Consider a material element of volume around the material particle **X**. When the body is in the reference state, the volume of the element is $dV(\mathbf{X})$. As the body deforms, the mass of the element remains unchanged, and is $\rho_R(\mathbf{X})dV(\mathbf{X})$ at all time. The linear momentum of any part of the body is

$$\int \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}\rho_R(\mathbf{X})dV(\mathbf{X}).$$

The integral extends over the volume of the part in the reference state. The rate of change of the linear momentum of the part is

$$\frac{d}{dt}\int \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}\rho_R(\mathbf{X})dV(\mathbf{X})=\int \frac{\partial^2 \mathbf{x}(\mathbf{X},t)}{\partial t^2}\rho_R(\mathbf{X})dV(\mathbf{X}).$$

The integrals extend over the volume of the part in the reference state.

Now change variable from **X** to **x**. Recall that

$$\rho_R(\mathbf{X})dV(\mathbf{X})=\rho(\mathbf{x},t)dv(\mathbf{x}),$$

and

$$\frac{\partial^2 x_i(\mathbf{X},t)}{\partial t^2}=\frac{\partial v_i(\mathbf{x},t)}{\partial t}+\frac{\partial v_i(\mathbf{x},t)}{\partial \mathbf{x}_j}v_j(\mathbf{x},t).$$

Consequently, the rate of change of linear moment of the material element is

$$\int\left[\frac{\partial v_i(\mathbf{x},t)}{\partial t}+\frac{\partial v_i(\mathbf{x},t)}{\partial \mathbf{x}_j}v_j(\mathbf{x},t)\right]\rho(\mathbf{x},t)dv(\mathbf{x}).$$

The integral now extends over the volume of the material element when the body is in the current state.

Conservation of linear momentum. In the current state at time t, consider a material particle at place **x**. Let $\mathbf{n}da$ be an element of area at **x**, where **n** is the unit vector normal to the element, and da is the area of the element. The force acting on this element is designated as $\mathbf{t}da$, where $\mathbf{t}(\mathbf{x},t)$ is the force acting on the element divided by the area of the element. Let dv be the element of volume at **x**, and $\mathbf{b}(\mathbf{x},t)dv$ be the force acting on the element.

Consider a part of the body. The conservation of moment requires that the force acting on the part equals the rate of change of the linear moment in the part:

$$\int t_i(\mathbf{x},t)da + \int b_i(\mathbf{x},t)dv = \int\left[\frac{\partial v_i(\mathbf{x},t)}{\partial t} + \frac{\partial v_i(\mathbf{x},t)}{\partial \mathbf{x}_j}v_j(\mathbf{x},t)\right]\rho(\mathbf{x},t)dv .$$

The integral extends over the part when the body is in the current state.

True stress. For a bar pulled by an axial force, the true stress is defined as the axial force in the current state divided by the cross-sectional area of the bar in the current state. We now generalize this definition to three dimensions.

Consider a small part of the body. When the body is in the current state, the small part is a rectangular block with faces parallel to the coordinate planes. Now consider one face of the block. The face is normal to the axis x_j. Acting on this face is a force. Denote by σ_{ij} the component *i* of the force acting on this face in the current state divided by the area of the face in the reference state. The nine quantities σ_{ij} are components of the true stress, or the Cauchy stress.

We adopt the following sign convention. When the outward normal vector of the area points in the positive direction of axis x_j, we take σ_{ij} to be positive if the component *i* of the force points in the positive direction of axis x_i. When the outward normal vector of the area points in the negative direction of the axis x_j, we take σ_{ij} to be positive if the component *i* of the force points in the negative direction of axis x_i.

Conservation of linear momentum in differential form. The law of the conservation of momentum requires that

$$\frac{\partial \sigma_{ij}(\mathbf{x},t)}{\partial x_j} + b_i(\mathbf{x},t) = \rho(\mathbf{x},t)\left[\frac{\partial v_i(\mathbf{x},t)}{\partial t} + \frac{\partial v_i(\mathbf{x},t)}{\partial \mathbf{x}_j}v_j(\mathbf{x},t)\right],$$

in the volume of the body, and

$$\sigma_{ij}n_j = t_i$$

on the surface of the body. These are familiar equations used in fluid mechanics.

Exercise. Start with the conservation of linear momentum in integral form, and prove the conservation of linear momentum in differential form.

Conservation of angular momentum. The law of the conservation of momentum requires that

$$\sigma_{ij} = \sigma_{ji} .$$

Exercise. Prove the above equation.

Exercise. Prove that

$$\frac{\text{work in the curent state}}{\text{volume in the } \textit{current} \text{ state}} = \sigma_{ij} \delta \varepsilon_{ij} .$$

Relation between true stress and nominal stress. Consider a material element of area. In the vector form, the element is $\mathbf{N}dA$ when the body is in the reference state, and the same element becomes $\mathbf{n}da$ when the body is in the current state. As proved before, the two vectors are related as

$$F_{iK} n_i da = \det(\mathbf{F}) N_K dA .$$

The two expressions, $t_i da$ and $T_i dA$, represent the same physical quantity: the force acting on the material element of area in the current state, so that $t_i da = T_i dA$. Recall that $\sigma_{ij} n_j = t_i$ and $s_{iK} N_K = T_i$, and we obtain that

$$\sigma_{ij} n_j da = s_{iK} N_K dA .$$

A combination of the above two equations gives that

$$\sigma_{ij} = \frac{s_{iK} F_{jK}}{\det(\mathbf{F})} .$$

An alternative derivation of the relation between true stress and nominal stress. Recall that

$$\frac{\text{work in the curent state}}{\text{volume in the reference state}} = s_{iK} \delta F_{iK} ,$$

and

$$\frac{\text{work in the curent state}}{\text{volume in the } \textit{current} \text{ state}} = \sigma_{ij} \delta \varepsilon_{ij} .$$

Equating the work on an element of material, we obtain that

$$\sigma_{ij} \delta \varepsilon_{ij} dv = s_{iK} \delta F_{iK} dV .$$

Because the true stress is a symmetric tensor, we obtain that

$$\sigma_{ij} \delta \varepsilon_{ij} = \sigma_{ij} \frac{\partial \delta u_i(\mathbf{x})}{\partial x_j} .$$

Also note that

$$\delta F_{iK} = \frac{\partial \delta U_i(\mathbf{X})}{\partial X_K} = \frac{\partial \delta u_i(\mathbf{x})}{\partial x_j}\frac{\partial x_j(\mathbf{X},t)}{\partial X_K} = \frac{\partial \delta u_i(\mathbf{x})}{\partial x_j} F_{jK}(\mathbf{X},t),$$

and that

$$dv / dV = \det(\mathbf{F}).$$

The two expressions for the work be the same for all motion. We obtain that

$$\sigma_{ij} = \frac{s_{iK} F_{jK}}{\det(\mathbf{F})}.$$

Exercise. For an incompressible and elastic material, confirm that

$$\sigma_{ij} = F_{jK}\frac{\partial W(\mathbf{F})}{\partial F_{iK}} - \Pi \delta_{ij}.$$

Newtonian fluids. The components of the stress relate to the components of the rate of deformation as

$$\sigma_{ij} = \eta\left[\frac{\partial v_i(\mathbf{x},t)}{\partial x_j} + \frac{\partial v_j(\mathbf{x},t)}{\partial x_i}\right] - p\delta_{ij},$$

where η is the viscosity. The material is assumed to be incompressible:

$$\frac{\partial v_i(\mathbf{x},t)}{\partial x_i} = 0.$$

Exercise. Derive the Navier-Stokes equation.

References

T. Belytschko, W.K. Liu and B. Moran, Nonlinear Finite Elements for Continua and Structures. Wiley, 2000.

A.F. Bower, Applied Mechanics of Solids. The full text is free online: http://solidmechanics.org/

M.E. Gurtin, E. Fried and Lallit Anand, The Mechanics and Thermodynamics of Continua. Cambridge University Press, 2010.

G.A. Holzapfel, Nonlinear Solid Mechanics, Wiley, 2000.

C. Trusdell and W. Noll, The Non-linear Field Theories of Mechanics, 3rd edition, Springer, 2004.

J.L. Wegner and J.B. Haddow, Elements of continuum mechanics and thermodynamics. Cambridge University Press, 2009.

匡震邦, 非线性连续介质力学, 上海交通大学出版社, 2002. Written by the man who taught me the subject in 1985 in Xian Jiaotong University.

黄筑平，连续介质力学基础，高等教育出版社， 2003.