Trouble with linear elastic theory of strength

Readings

- Z. Suo, Elements of linear elasticity (http://imechanica.org/node/205).
- C.R. Kurkjian and U.C. Paek, "Single-valued strength of perfect silica fibers," Appl. Phys. Lett. 42, 251-253 (1983).

The problem of fracture. A body is subject to a load. What is the magnitude of the load that will cause the body to fracture? Let us begin with a body made of a silica glass, which deforms elastically by small strains. A procedure you have been taught before probably goes as follows. You first determine the maximum stress in the body. You then determine the strength of the material. The body is supposed to fracture when the maximum stress in the body reaches the strength of the material.

I'll first review this procedure, so that you and I agree exactly what this procedure is. I'll then explain why this procedure is difficult to apply in practice.

Theory of linear elasticity. I assume that you know the rudiments of the theory of linear elasticity. You know, for example, that the **state of stress** of a material particle is described by a **tensor**, and that the state of stress in a body is described by a **field**. Furthermore, the field of stress in the body may be determined by solving a boundary-value problem. The theory of elasticity can be found in many textbooks. You can find a list of the governing equations of elasticity in my notes (http://imechanica.org/node/205). I'll not spend much time on these equations in class. I will, however, slow down whenever features of the theory of elasticity play significant roles in developing fracture mechanics.

A few quick notes about solving boundary-value problems in elasticity:

- Governing equations. Quickly list them in class.
- Boundary conditions. Prescribe traction. Prescribe displacement.
- Approximate analytical solutions. Beams and plates.
- Exact analytical solutions. Few problems can be solved exactly. Timoshenko and Goodier describe some.
- Handbooks contain useful solutions obtained by the above methods.
- Numerical solutions. Finite element method. Commercial software such as ABAQUS. A local community of users.

Maximum stress in a body. Imagine a body (e.g., a plate with a hole) subject to an applied stress, σ_{appl}. At each material particle, the state of stress is a tensor, with 6 components. In the body, the state of stress varies form one material particle to another. Thus, the state of stress in the body is described by a field. This field is determined by solving the boundary-value problem. Solving boundary-value problems is a big task by itself, but is not the subject of this course. Let's say we already have the solution. That is, we know all six components of stress at every material particle in the body, $\sigma_{11}(x_1,x_2,x_3)$, $\sigma_{12}(x_1,x_2,x_3)$…

What do we do with this massive amount of data? We are interested in predicting the conditions of fracture of the body. For example, if the applied stress σ_{appl} is small enough, the body will not fracture. How small is small enough?

Here is a procedure taught in the theory of elasticity. From the field of stress we determine the maximum stress in the body, $\sigma_{\max}$. The maximum component of stress at each material particle is determined by a principal stress, solved by an eigenvalue problem. Then we look for the largest value of the principal stress by comparing all material particles in the body.

Today, all this procedure is embodied in commercial software such as ANAQUS. So you know the maximum component of stress in the body, $\sigma_{\max}$.

Stress concentration factor. The equations in elasticity are linear, so that the maximum stress in the body is proportional to the applied stress. We write

$$\sigma_{\max} = C\sigma_{appl} ,$$

where C is a dimensionless number. The basic phenomenon that the stress is higher at some material particles in a body than others is known as stress concentration. The number C is known as the stress concentration factor.

When the hole is circular, and the plate is much larger than the hole, the boundary-value problem is solved analytically in Timoshenko and Goodier. The maximum stress occurs at the surface of the hole. The stress concentration factor is

$$\frac{\sigma_{\max}}{\sigma_{appl}} = 3 .$$

Two significant features of linear elasticity. First, because the governing equations of elasticity are linear, the stress concentration factor is independent of the applied stress. Second, because the governing equations of elasticity contain no length, the stress concentration factor depends on ratios of the lengths in boundary conditions.

For a circular hole in an infinite plate, the only length in the boundary conditions is the radius of the hole. Consequently, the stress concentration factor is a dimensionless number, independent of any parameter. In particular, the stress concentration factor of the circular hole is independent of the radius of the hole.

Elliptic hole. As another example of boundary-value problems, consider an elliptic hole in an infinite plate subject to a remote stress σ_{appl}. The problem can still be solved analytically (Inglis, 1913). The maximum stress $\sigma_{\max}$ in the plate is given by

$$\frac{\sigma_{\max}}{\sigma_{appl}} = 1 + \frac{2a}{b} ,$$

where a and b are semi-axes of the ellipse. The stress concentration factor depends on the shape of the hole, characterized by the ratio a/b. When $a = b$, the hole is circular, and the stress concentration factor is 3. When the ellipse is very elongated, $a \gg b$, the stress concentration factor is very large.

The problem of the elliptic hole would have been yet another unremarkable boundary-value problem were not for an inspired way of using its solution. You might say that 100 years ago it was so difficult to solve boundary-value problems that whenever a particular problem is solved you would try to make the most out of the solution. Here is the inspired, if not dubious, way of using the solution of the elliptic hole. Do some math and convince yourself that the radius of curvature at the tip of the ellipse is $\rho = b^2 / a$. You can express the above formula in terms a and ρ, namely,

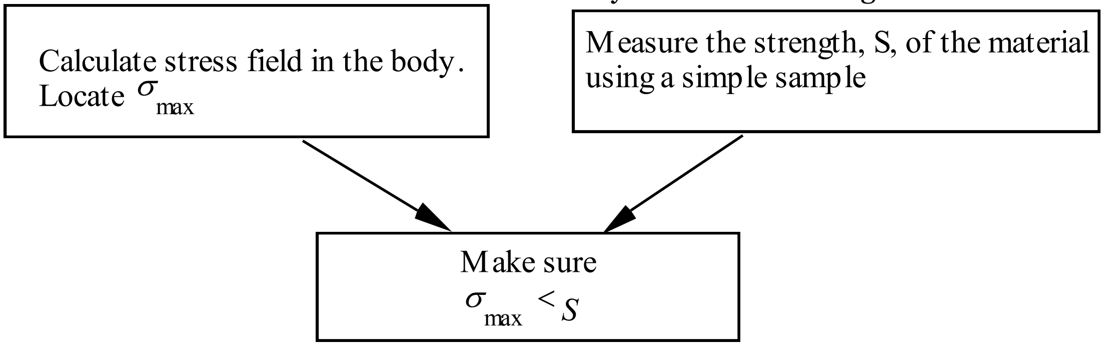
$$\frac{\sigma_{\max}}{\sigma_{appl}} \approx 2\left(\frac{a}{\rho}\right)^{1/2} .$$

This formula may be used to estimate the stress concentration factor for a flaw of some other shape, where a is interpreted as the “overall size” of the flaw, and ρ is the radius at the root of the flaw.

Strength of a material. Now we have calculated the maximum stress $\sigma_{\max}$ in the body. Will the body sustain this stress? The theory of elasticity will *not* answer this question. Nothing, absolutely nothing, in the equations of elasticity will answer this question. You need to find the strength of the material from somewhere else. For example, you can determine the strength of the material experimentally. You pull a sample until it breaks. You record the stress that breaks the sample. You call this stress the strength of the material, S. Because it is determined experimentally, we will call it the experimental strength. The experimental values for glass are on the order $S \sim 100$ MPa.

Design for strength based on linear elasticity. What is the maximum load that can be sustained by a body? We can now summarize the procedure as follows.

- Calculate the stress field by solving boundary-value problem. Locate the maximum stress $\sigma_{\max}$ in the body.
- Assume that the material has a definite strength. That is, the same material has the same strength, independent of the shape of the body. Measure the strength S using a simple sample of the material, such as a tensile bar.
- Make sure that the maximum stress in the body is below the strength of the material.



Why is the procedure of linear elasticity hard to use in practice?

(1) The maximum stress in a body is sensitive to the shape of the flaw.
(2) The shape of the flaw in the body is seldom known in practice.
(3) The procedure assumes that the body is linearly elastic everywhere, which is never true.
(4) The procedure assumes that the strength of a material is independent of the sample used in experiment. In reality, strengths measured from different samples are different, because each sample has different flaws.

That is, the procedure is hit in both ways: the maximum stress is impossible to calculate, and the strength is impossible to measure.

Theoretical Strength. For a body to fracture, the stress must break atomic bonds. The theoretical strength of a material can be calculated by atomistic simulation. A rough estimate is

$$S_{th} = \frac{E}{10}.$$

For a glass, Young's modulus is 70 GPa, so that the estimated theoretical strength is 7 GPa. This value is about two orders of magnitude higher than the strength measured in a bulk sample.

One exceptional case. Experimental strength for optic fibers. When a silica fiber develops etch pits, the stress concentration reduces the measured strength to

$$S_{\exp} = \frac{S_{th}}{C},$$

where C is the stress concentration factor. For an etch pit comparable to a hemisphere, $C = 2 \sim 3$. See the reading listed in the beginning of this lecture.

Experimental Strength for bulk samples. Now we have described two observations:

- For a typical macroscopic sample, the experimental strength is about 2 orders of magnitude lower than the theoretical strength.
- The experimental strength varies from one sample to another.

Both observations can be explained by a single idea: the strength of a sample depends on flaws in the sample. Specifically, let us apply the Inglis formula $\sigma_{\max} / \sigma_{appl} \approx 2(a/\rho)^{1/2}$ to a deep, sharp flaw (i.e., large a/ρ ratio). Take atomic dimension $\rho = 10^{-10}\mathrm{m}$, and flaw size $a = 10^{-6}\mathrm{m}$. We get $\sigma_{\max} / \sigma_{appl} \approx 200$. It is easy to understand the discrepancy between the theoretical strength and the experimental strength.

Body, flaws, and atoms. This little calculation suggests that perhaps there isn't any deep mystery about strength after all. All we really need to do is to calculate stress carefully. But to do so is really hard. The calculation involves at least three very different sizes: body, flaws, and atoms. In today's language, such a problem is known as a multi-scale problem.

In the above estimate, we have made several assumptions. In closer examination of these assumptions, we may convince ourselves that we've probably get the essential idea right: the strength of a body of glass can be knocked down by several orders of magnitude by tiny flaws. However, we cannot be sure that this procedure can give us values of strength reliable enough for engineering design.

Or maybe you would say, "Let us calculate stress more accurately by using more powerful computers, and more detailed atomistic model". Perhaps this brute-force approach will become viable in future. But not today. Certainly not at the time when Griffith came up with his ideas. We'll describe Griffith's approach to fracture in the next lecture.

The confrontation between Timoshenko and Swain. When I was a graduate student, a large portrait of George Swain could be found on the third floor of Pierce Hall. He was a professor of Engineering at Harvard from 1881 to 1927. I was told of a confrontation between him and Timoshenko. Searching on the Internet this morning, I found quite a few entries about the confrontation, some quite inflammatory, but I did not find a clear description of the technical issue that leads to the confrontation. (Google yourself for Swain Timoshenko.) In any event, here is what I heard. The issue concerned the stress concentration factor. It was said that Swain did not believe that the stress concentration factor of a circular hole is independent of the radius of the hole. He would say, a very small hole should have negligible effect on strength.

It is unimportant for us who said what. Let's simply focus on the effect of the size of the hole on the strength of a body. Indeed, the stress concentration factor of 3 is a result of linear elasticity. The result is correct so long as the assumption of linear elasticity is correct. In particular, the body has to be linearly elastic. This assumption can be violated, for example, by metals undergoing plastic deformation. Even for a brittle solid such as a silica glass, when the hole approaches atomic dimension, linear elastic assumption breaks down. It is entirely possible that a small enough hole will not reduce the breaking stress of a body. We will return to this question later in the course. But already you can tell that you should not trust linear elasticity.